Université Aix-Marseille

Introduction à l'analyse

Parcours PEIP

Planche 3ters Primitives

Déterminer une primitive des fonctions suivantes :

1. $f_1(x) = (3x+2)(x^3+2x^2+1) + e^{3x}$;
2. $f_2(x) = (3x^2+4x)(x^3+2x^2+1) + 2xe^{x^2+1}$;
3. $f_3(x) = \cos^6(x)\sin^4(x)$;
4. $f_4(x) = \cos^3(x)\sin^4(x)$;
5. $f_5(x) = e^x\cos(2x)$;
6. $f_6(x) = \frac{1}{\sqrt{1-9x^2}}$;
7. $f_7(x) = \frac{1}{4x^2-8x+6}$;
8. $f_8(x) = \frac{4+5x^2}{1+x^2}$;
9. $f_9(x) = \frac{x^3}{1-x^2}$;
10. $f_{10}(x) = \frac{1-x^3}{1-x^2}$;
11. $f_{11}(x) = \frac{1+x^3}{1-x^2}$;
12. $f_{12}(x) = \frac{3}{(x-1)(2x+1)}$;
13. $f_{13}(x) = \frac{1+2x}{(x-1)(2x^2+1)}$;
14. $f_{14}(x) = \frac{(1+x^2)^2-x}{x^3-x}$;
15. $f_{15}(x) = \arctan(x)$;
16. $f_{16}(x) = \arcsin(x)$;
17. $f_{17}(x) = \cos(x)\mathrm{argth}\big(\sqrt{2}\sin(x)\big)$;
18. $f_{18}(x) = \sqrt{1+x^2}$.

Solutions : dans ce qui suit, on note F_k une primitive de f_k pour tout entier k. Toute autre primitive sera obtenue en rajoutant une constante.

1. $f_1(x) = 3x^4 + 8x^3 + 4x^2 + 3x + 2 + e^{3x} \rightsquigarrow F_1(x) = \frac{3}{5}x^5 + 2x^4 + \frac{4}{3}x^3 + \frac{3}{2}x^2 + 2x + \frac{1}{3}e^{3x}$;
2. $f_2(x) = g_2'(x)g_2(x) + h_2'(x)e^{h_2(x)}$ avec $g_2(x) = x^3 + 2x^2 + 1$ et $h_2(x) = x^2 + 1 \rightsquigarrow F_2(x) = \frac{(x^3+2x^2+1)^2}{2} + e^{x^2+1}$;
3. $f_3(x) = \frac{(e^{ix}+e^{-ix})^6}{64}\frac{(e^{ix}-e^{-ix})^4}{16} = \frac{1}{1024}(e^{2ix} - e^{-2ix})^4(e^{ix} + e^{ix})^2 = \frac{1}{1024}(e^{8ix} - 4e^{4ix} + 6 - 4e^{-4ix} + e^{-8ix})(e^{2ix} + 2 + e^{-2ix}) = \frac{1}{1024}(e^{10ix} + 2e^{8ix} - 3e^{6ix} - 8e^{4ix} + 2e^{2ix} + 12 + 2e^{-2ix} - 8e^{-4ix} - 3e^{-6ix} + 2e^{-8ix} + e^{-10ix}) = \frac{1}{512}\big(\cos(10x) - 2\cos(8x) - 3\cos(6x) - 8\cos(4x) + 2\cos(2x) + 6\big) \rightsquigarrow F_3(x) = \frac{\sin 10x}{5120} + \frac{\sin(8x)}{2048} - \frac{\sin(6x)}{1024} - \frac{\sin(4x)}{256} + \frac{\sin(2x)}{512} + \frac{3x}{256}$;
4. $f_4(x) = f_3(x) = \cos(x)\big(1 - \sin^2(x)\big)\sin^4(x) = \cos(x)\sin^4(x) - \cos(x)\sin^6(x) \rightsquigarrow F_3(x) = \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7}$;
5. $f_5(x) = \Re(e^{(1+2i)x}) \rightsquigarrow F_5(x) = \Re\big(\frac{e^{(1+2i)x}}{1+2i}\big) = \Re\big(\frac{e^x(1-2i)e^{2ix}}{5}\big) = e^x\left(\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5}\right)$;
6. $f_6(x) = \frac{1}{3}\frac{3}{\sqrt{1-(3x)^2}} \rightsquigarrow F_6(x) = \frac{1}{3}\arcsin(3x)$;
7. $f_7(x) = \frac{1}{(2x-2)^2-4+6} = \frac{1}{(2x-2)^2+2} = \frac{1}{2}\frac{1}{\big(\sqrt{2}(x-1)\big)^2+1} = \frac{1}{2\sqrt{2}}\frac{\sqrt{2}}{\big(\sqrt{2}(x-1)\big)^2+1} \rightsquigarrow F_7(x) = \frac{1}{\sqrt{2}}\arctan\big(\sqrt{2}(x-1)\big)$;
8. $f_8(x) = \frac{5+5x^2}{1+x^2} - \frac{1}{1+x^2} = 5 - \frac{1}{1+x^2} \rightsquigarrow F_8(x) = 5x + \arctan(x)$;
9. $f_9(x) = \frac{x^3-x+x}{1-x^2} = -x + \frac{x}{1-x^2} \rightsquigarrow F_9(x) = -\frac{x^2}{2} - \frac{1}{2}\ln|1-x^2|$;
10. $f_{10}(x) = \frac{(1-x)(1+x+x^2)}{(1-x)(1+x)} = \frac{1+(1+x)x}{1+x} = x + \frac{1}{1+x} \rightsquigarrow F_{10}(x) = \frac{x^2}{2} + \ln|1+x|$;
11. $f_{11}(x) = \frac{1+x(x^2-1)+x}{1-x^2} = -x + \frac{1+x}{1-x^2} = -x - \frac{1}{x-1} \rightsquigarrow F_{11}(x) = -\frac{x^2}{2} - \ln|1-x|$;
12. $f_{12}(x) = \frac{A}{x-1} + \frac{B}{2x+1}$ pour certains $A, B \in \mathbb{R}$. Par identification, on trouve $f_{12}(x) = \frac{1}{x-1} - \frac{2}{2x+1} \rightsquigarrow F_{12}(x) = \ln|x-1| - \ln|2x+1| = \ln\left|\frac{x-1}{2x+1}\right|$;
13. $f_{13}(x) = \frac{A}{x-1} + \frac{B+Cx}{2x^2+1}$ pour certains $A, B, C \in \mathbb{R}$. Par identification, on trouve $f_{13}(x) = \frac{1}{x-1} - \frac{2x}{2x^2+1} \rightsquigarrow F_{13}(x) = \ln|x-1| - \frac{1}{2}\ln(2x^2+1) = \ln\left|\frac{x-1}{\sqrt{2x^2+1}}\right|$;
14. $f_{14}(x) = \frac{1+2x^2+x^4-x}{x^3-x} = \frac{1-x+2x^2+x^4}{x^3-x} = \frac{1-x+2x^2+x(x^3-x)+x^2}{x^3-x} = x + \frac{1-x+3x^2}{x(x^2-1)} = x + \frac{A}{x} + \frac{B+Cx}{x^2-1}$ pour certains $A, B, C \in \mathbb{R}$. Par identification, on trouve $f_{14}(x) = x - \frac{1}{x} - \frac{1}{x^2-1} + \frac{4x}{x^2-1} \rightsquigarrow F_{14}(x) = \frac{x^2}{2} - \ln|x| + \operatorname{argth}(x) + 2\ln|x^2-1| = \frac{x^2}{2} + \operatorname{argth}(x) + \ln\left|\frac{(x^2-1)^2}{x}\right|$;
15. $F_{15}(x) = \int_0^x \arctan(t)dt = [t.\arctan(t)]_0^x - \int_0^t \frac{tdt}{1+t^2}$ en intégrant par partie avec $u'(t) = 1$ et $v(t) = \arctan(t)$. On a donc $F_{16}(x) = x.\arctan(x) - \frac{1}{2}[\ln(1+t^2)]_0^x = x.\arctan(x) - \ln(\sqrt{1+x^2})$;
16. $F_{16}(x) = \int_0^x \arcsin(t)dt = [t.\arcsin(t)]_0^x - \int_0^t \frac{tdt}{\sqrt{1-t^2}}$ en intégrant par partie avec $u'(t) = 1$ et $v(t) = \arcsin(t)$. On a donc $F_{16}(x) = x.\arcsin(x) + [\sqrt{1-t^2}]_0^x = x.\arcsin(x) + \sqrt{1-x^2} - 1$;
17. $F_{17}(x) = \int_0^x \cos(t)\operatorname{argth}\big(\sqrt{2}\sin(t)\big)dt = [\sin(t)\operatorname{argth}\big(\sqrt{2}\sin(t)\big)]_0^x - \int_0^x \sin(t)\frac{\sqrt{2}\cos(t)}{1-2\sin^2(t)}$ en intégrant par partie avec $u'(t) = \cos(t)$ et $v(t) = \operatorname{argth}\big(\sqrt{2}\sin(t)\big)$. On a donc $F_{17}(x) = \sin(x)\operatorname{argth}\big(\sqrt{2}\sin(x)\big) - \int_0^x \frac{2\sin(t)\cos(t)}{\sqrt{2}\cos(2t)} = \sin(x)\operatorname{argth}\big(\sqrt{2}\sin(x)\big) - \frac{1}{\sqrt{2}}\int_0^x \frac{\sin(2t)}{\cos(2t)} = \sin(x)\operatorname{argth}\big(\sqrt{2}\sin(x)\big) + \frac{1}{2\sqrt{2}}\big[\ln\big|\cos(2x)\big|\big]_0^x = \sin(x)\operatorname{argth}\big(\sqrt{2}\sin(x)\big) + \frac{1}{2\sqrt{2}}\ln\big(\cos(2x)\big)$;
18. $F_{18}(x) = \int_0^x \sqrt{1+t^2}dt = [t\sqrt{1+t^2}]_0^x - \int_0^x \frac{t^2dt}{\sqrt{1+t^2}}$ en intégrant par partie avec $u'(t) = 1$ et $v(t) = \sqrt{1+t^2}$. On a donc $F_{18}(x) = x\sqrt{1+x^2} - \int_0^x \frac{(1+t^2-1)dt}{\sqrt{1+t^2}} = x\sqrt{1+x^2} - \int_0^x \sqrt{(1+t^2)}dt + \int_0^x \frac{dt}{\sqrt{1+t^2}} = x\sqrt{1+x^2} - F_{18}(x) + [\operatorname{argsh}(t)]_0^x$. Cela donne $2F_{18}(x) = x\sqrt{1+x^2} + \operatorname{argsh}(x)$ et donc $F_{18}(x) = \frac{1}{2}\big(x\sqrt{1+x^2} + \operatorname{argsh}(x)\big)$