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Equi-coercivity of variational problems: the role of recession functions

Abstract: *This paper deals with the coercivity of functionals under the form $F - \lambda L$ where F is a convex and coercive function, L is a linear form and λ is a scalar parameter. The equi-coercivity of a sequence $F^\epsilon - \lambda L$, where F^ϵ is equi-coercive and converges to F, is considered. These questions are shown to be strongly related to the variational problem* Inf $\{F_\infty(u),\ L(u) = 1\}$ *where F_∞ is the recession function of F. Convergence of recession functions is also examined. Several applications to plasticity and capillarity are given.*

1. Outline of the paper

This paper deals with variational problems modelling several physical situations which can be posed in the form:

$$\operatorname*{Inf}_{u \in X} \ \{F(u) - \lambda L(u)\}, \tag{1.1}$$

where X is a Banach space, F is a convex functional, L is a linear form and λ is a scalar parameter. We look for the values of λ for which the Infimum in (1.1) is finite and reached. A well known sufficient condition ensuring that the infimum is finite is the coercivity of the functional $F - \lambda L$ at infinity with respect to $\|u\|$, in a sense specified later. The case of a quadratic F is standard and the growth of $F - \lambda L$ at infinity is governed by the growth of F only. In the cases of special interest for us, F has only a linear growth with respect to $\|u\|$ at infinity and the behaviour of $F - \lambda L$ for large $\|u\|$ is not a straightforward consequence of that of F.

Typical examples of this situation are provided by capillarity, quasi-linear equations under nonlinear boundary conditions and plasticity. Let us precise the latter case which has important applications in Mechanics. The equilibrium problem of an elasto-plastic body obeying Hencky's law of plasticity and submitted to loads proportional to a scalar parameter λ, can be formulated in term of displacements as a minimization problem:

$$\operatorname*{Inf}_{u = 0 \text{ on } \Gamma_0} \ \left\{ \int_\Omega j(x, e(u))\, dx - \lambda L(u) \right\}. \tag{1.2}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $u: \Omega \to \mathbb{R}^N$ is a vector valued field (rate of displacement), $e(u)$ is its deformation tensor, Γ_0 is the subset of the boundary $\partial\Omega$ where the body is fixed. $j(x,E)$ is convex with respect to E and satisfies:

$$k_0\left(|E^D| + |trE|^2 - 1\right) \le j(x,E) \le k_1\left(|E^D| + |trE|^2 + 1\right), \qquad (1.3)$$

where trE and E^D are the trace and the deviatoric part of E (notations will be specified in section 3). It is important to note that j has only a linear growth at infinity with respect to E (see the term $|E^D|$). The linear form L reads as:

$$L(u) = \int_\Omega f_0 u\ dx + \int_{\Gamma_1} g_0 u\ ds. \qquad (1.4)$$

where f_0 are the body forces and g_0 are the surface loads on $\Gamma_1 = \partial\Omega - \Gamma_0$.

The discussion of the limit values of the load parameter λ for which the infimum in (1.1) is finite, is known as the limit load problem and has been widely discussed in the literature (TEMAM [15] for a mathematical account and SALENCON [14] for a mechanical point of view).

The aim of the present paper is to discuss variational problems such as (1.1) with weak growth condition. In section 2 we discuss, for a coercive functional F, the coercivity of the functional $F - \lambda L$. Under specific assumptions, the coercivity of $F - \lambda L$, where λ is taken to be positive in order to fix ideas, is ensured for any λ strictly smaller than $\bar{\lambda}$:

$$\lambda < \bar{\lambda} = \text{Inf}\,\{F_\infty(u), \quad L(u) = 1\}. \qquad (1.5)$$

F_∞ is the recession function of F defined in section 2. A similar inequality holds for negative λ's.

In section 3 we discuss the relation between coercivity and Γ-convergence. More specifically let $(F^\epsilon)_{\epsilon>0}$ be a sequence of convex functionals converging to F, then the equi-coercivity of the sequence $(F^\epsilon - \lambda L)_{\epsilon>0}$ is ensured by (1.5). To illustrate the objective of this section, consider the variational problem (1.2) arising in plasticity, where $j(x,E)$ is replaced by an ϵ-periodic (with respect to the variable x) function $j^\epsilon(x,E)$:

$$\underset{u=0 \text{ on } \Gamma_0}{\text{Inf}} \left\{\int_\Omega j^\epsilon(x,e(u))\ dx - \lambda L(u)\right\}. \qquad (1.6)$$

When ϵ goes to 0, (1.6) is a sequence of variational problems describing the equilibrium of nonhomogeneous elasto-plastic composites, ϵ-periodic assembly of different constituents. The question answered in section 3 is the determination of the values of λ for which the infimum in (1.6) is finite for ϵ small enough, by means of an homogenized problem.

Section 4 deals with the Γ-convergence of recession functions. Consider a sequence of functionals $(F^\epsilon)_{\epsilon>0}$ converging to F. We establish that, under appropriate hypotheses, the sequence of recession functions F^ϵ_∞ converges to F_∞. When applied to the functional of plasticity (limit load problem) our result gives directly an homogenization result for incompressible media. It does not require any approximation procedure for admissible functions satisfying the constraint $\text{div}(u) = 0$.

2. Coercive Functionals

2.1. The main result.

• In the sequel $(X, \|\cdot\|)$ will denote a Banach space, which can be endowed with a topology τ such that:

$$\textit{closed balls in } (X, \|\cdot\|) \textit{ are } \tau - \textit{compact}. \qquad (2.1)$$

Typical examples of this situation considered in the sequel are:

a) X is a reflexive space (often $W^{1,p}(\Omega)$ with $p > 1$), τ is the weak topology on X.

b) $X = BV(\Omega)$ or $BD(\Omega)$ (see definitions of these spaces in (3.16) (3.17)), τ is the L^p topology $(1 < p < (N/(N-1))$.

• $F: X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper, convex, τ-lower semi-continuous (τ-l.s.c. in abbreviated form) functional on X. F_∞ denotes its recession function defined as:

$$F_\infty(u) = \underset{t>0}{\text{Sup}}\ \frac{1}{t}F(tu + u_0), \quad \text{where } u_0 \in \text{dom}(F). \qquad (2.2)$$

The definition of F_∞ does not depend on the choice of u_0 and the supremum in (2.2) can be replaced by $\lim_{t \to +\infty}$

• Among all possible definitions of a coercive functional (ATTOUCH [1], BAIOCCHI *et al* [2]) we shall choose the following one:

Definition 1. *$F: X \to \overline{\mathbb{R}}$ is coercive iff:*

$$\lim_{\|u\| \to +\infty} F(u) = +\infty.$$

A few simple relations between properties of F and F_∞ will be useful in the sequel:

Proposition 1. *Let $F: X \to \overline{\mathbb{R}}$ be a proper, convex and τ-lower semi-continuous functional. Then:*

(i) F_∞ *is proper, convex and* τ*-lower semi-continuous.*

(ii) $F_\infty^* = \mathbf{I}_{\mathrm{cl-dom}(F^*)}$ *and :*

$$\forall\, u, v \in X \times X:\ F(u+v) \le F(u) + F_\infty(v). \tag{2.4}$$

(iii) $\operatorname*{Inf}_{u \in X} F(u) > -\infty \Rightarrow F_\infty \ge 0.$

Equivalence holds under the additional assumption that $\mathrm{dom}(F^*)$ *is closed.*

(iv) If F *is coercive then* F_∞ *is coercive,* $F_\infty(u) > 0$ *for* $u \ne 0$, *and:*

$$-\infty < \operatorname*{Inf}_{u \in X} F(u) = \operatorname*{Min}_{u \in X} F(u) < +\infty.$$

Comment: The following notations have been used:

$$\mathrm{dom}(F) = \{u \in X,\ F(u) < +\infty\}, \quad \mathrm{cl-dom}(F^*) = \overline{\mathrm{dom}(F^*)},$$

$$\mathbf{I}_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{otherwise} \end{cases}$$

Proof of Proposition 1: Points i) and ii) are classical (see e.g. [8]).

In order to prove iii), note that:

$$\operatorname*{Inf}_{u \in X} F(u) = -F^*(0).$$

Therefore

$$\operatorname*{Inf}_{u \in X} F(u) > -\infty \ \Leftrightarrow\ 0 \in \mathrm{dom}(F^*) \ \Leftrightarrow\ \mathbf{I}_{\mathrm{dom}(F^*)} \le \mathbf{I}_{\{0\}}$$

$$\Rightarrow (\mathbf{I}_{\mathrm{dom}(F^*)})^* \ge (\mathbf{I}_{\{0\}})^* \Leftrightarrow F_\infty \ge 0.$$

Equivalence holds under the additional assumption that $\mathrm{dom}(F^*)$ is closed.

In order to prove iv) note that, according to (2.4):

$$\forall u_0 \in \mathrm{dom}(F), \quad F_\infty(u) \ge F(u+u_0) - F(u_0),$$

and therefore that F_∞ satisfies (2.3) and is coercive. The coercivity of F implies

$$\operatorname*{Inf}_{u \in X} F(u) > -\infty.$$

Indeed it is deduced from the coercivity condition that:

$$\operatorname*{Inf}_{u \in X} F(u) = \operatorname*{Inf}_{u \in B} F(u),$$

where B denotes a closed ball of sufficient radius in X. By virtue of (2.1), B is a τ-compact set on which the τ-lower semi-continuous functional F attains its infimum which is therefore a minimum. This minimum value is finite since F is proper. It remains to prove that $F_\infty(u) > 0$ for $u \ne 0$. Note that for every u and u_0 in X and every positive t, ii) together with the positive homogeneity of degree one of F_∞ yields:

$$F_\infty(u) \ge \frac{F(u_0 + tu) - F(u_0)}{t}.$$

If $F_\infty(u) = 0$, we obtain:

$$F(u_0 + tu) \le F(u_0) \quad \forall t > 0,\ \forall u_0 \in X.$$

This last inequality is in contradiction with the coercivity and the properness of F. This completes the proof of Proposition 1.

The main result of this section is the following:

Theorem 1. *Let* $(X, \|\,.\,\|)$ *and* τ *satisfy* (2.1). *Let* F *be a proper, convex, coercive functional defined on* X, *and* τ*-lower semi-continuous. Let* L *be a linear* τ*-continuous form on* X*. Then i) and ii) are equivalent:*

i) $F - \lambda L$ *is coercive,*

ii) $\underline{\lambda} < \lambda < \overline{\lambda}$ *where :*

$$\overline{\lambda} = \mathrm{Min}\,\{F_\infty(u),\ u \in X,\ L(u) = 1\},$$

$$\underline{\lambda} = -\mathrm{Min}\,\{F_\infty(u),\ u \in X,\ L(u) = -1\}.$$

Comments: Theorem 1 is a particular case of Theorem 2 below, and will not be proved here, although a direct proof can be given. We just comment on the fact that the definitions of $\overline{\lambda}$ and $\underline{\lambda}$ contain a Min and not an Inf. Indeed it can be deduced from Proposition 1 that F_∞ is a proper, convex, τ-l.s.c. and coercive functional. Since L is τ-continuous the functional

$$G = F_\infty + \mathbf{I}_{\{L(u)=1\}},$$

is also convex, coercive and τ-l.s.c. (but not necessarily proper). Therefore its minimum value $\overline{\lambda}$ is reached.

2.2. Examples. *1. Quasi-linear equations.*

Consider the following problem on a bounded open set Ω of $\mathbb{R}^N$, with a regular boundary $\partial\Omega$:

$$\left.\begin{array}{l} f \in -\Delta u + \beta(u) \text{ in } \Omega \\ \\ g \in \dfrac{\partial u}{\partial n} + \gamma(u) \text{ on } \partial\Omega \end{array}\right\} \tag{2.5}$$

where f is in $L^2(\Omega)$, g is in $L^2(\partial\Omega)$, β and γ are two maximal monotone graphs on $\mathbb{R}$ exhibiting the following properties:

$$0 \in \beta(0), \quad 0 \in \gamma(0), \quad 0 \text{ is an interior point of } \overline{\mathrm{Im}(\beta)}. \tag{2.6}$$

A weak solution of (2.5) is searched:

i) $\Delta u + f \in L^1(\Omega)$ and $\Delta u(x) + f(x) \in \beta(u(x))$ a.e. $x \in \Omega$,

ii) $g - \dfrac{\partial u}{\partial n}$ is a $L^1(\partial\Omega)$ selection of $\gamma(u)$

(for the definition of $\dfrac{\partial u}{\partial n}$ in the sense of distributions see DAUTRAY & LIONS [9] ch. 2 p. 583).

Define in $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$:

$$[\beta_-, \beta_+] = \overline{\mathrm{Im}(\beta)}, \quad [\gamma_-, \gamma_+] = \overline{\mathrm{Im}(\gamma)},$$

and let j_β and j_γ be primitives of β and γ vanishing at 0. Then j_β and j_γ are convex, l.s.c. and non negative functions. The variational form of (2.5) reads:

$$\operatorname*{Inf}_{u \in H^1(\Omega)} \{F(u) - L(u)\}, \tag{2.7}$$

where:

$$F(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\, dx + \int_\Omega j_\beta(u)\, dx + \int_{\partial\Omega} j_\gamma(u)\, ds\,,$$

$$L(u) = \int_\Omega fu\, dx + \int_{\partial\Omega} gu\, ds.$$

$X = H^1(\Omega)$, τ is the weak topology. It is straightforward to check that L is τ-continuous and F is proper, convex, τ-l.s.c. and coercive. A sufficient condition ensuring that the variational problem (2.7) possesses a solution is that $F - L$ is coercive, which, according to Theorem 1, is equivalent to:

$$1 < \operatorname*{Inf}_u \{F_\infty(u), \quad L(u) = 1\}. \tag{2.8}$$

It is readily seen that F_∞ reads:

$$F_\infty(u) = \begin{cases} \int_\Omega (j_\beta)_\infty(u)\, dx + \int_{\partial\Omega} (j_\gamma)_\infty(u)\, ds & \text{if } u \text{ is constant on } \Omega \\ \\ +\infty & \text{otherwise} \end{cases}$$

(2.8) is hence equivalent to a condition already derived by different means in BENILAN *et al* [3] :

$$\beta_-|\Omega| + \gamma_-|\partial\Omega| < \int_\Omega f\, dx + \int_{\partial\Omega} g\, ds < \beta_+|\Omega| + \gamma_+|\partial\Omega|.$$

2. Capillarity problem.

The τ-continuity of L and the τ-lower semi-continuity of F cannot be removed from the assumptions of Theorem 1. The capillarity problem provides a counter example when L is not τ-continuous. Section 3 contains a discussion of the modifications to bring to Theorem 1 to cover the case where F is not τ-l.s.c..

The determination of a liquid free surface as the resultant of surface forces, gravity forces and boundary adhesion, can be reduced in many situations to the following variational problem on a bounded open set Ω of $\mathbb{R}^N$ with a Lipschitz boundary:

$$\operatorname*{Inf}_{u \in W^{1,1}(\Omega)} \left\{ \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \frac{k}{2} \int_\Omega |u|^2\, dx - \lambda \int_{\partial\Omega} u\, ds \right\}. \tag{2.11}$$

When $\lambda = 0$, minimizing sequences of (2.11) are bounded in $W^{1,1}(\Omega)$, hence also in $L^{N/(N-1)}(\Omega)$. A natural choice for the pair (X, τ) is consequently $X = W^{1,1}(\Omega)$, τ = weak topology of $L^{N/(N-1)}(\Omega)$, and :

$$F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \frac{k}{2} \int_\Omega |u|^2\, dx, \quad L(u) = \int_{\partial\Omega} u\, ds. \tag{2.12}$$

Note that L is not τ-continuous. A straightforward computation leads to (with $k > 0$):

$$F_\infty(u) = \begin{cases} 0 & \text{if } u = 0 \text{ a.e. in } \Omega, \\ \\ +\infty & \text{otherwise} \end{cases}$$

Therefore:

$$\operatorname*{Inf}_u \{F_\infty(u),\ L(u) = 1\} = \operatorname*{Inf}_u \{F_\infty(u),\ L(u) = -1\} = +\infty,$$

and an erroneous use of Theorem 1 in this case would imply that $F-\lambda L$ is coercive for any value of λ, while it is well known (MASSARI and MIRANDA [13]) that $\mathrm{Inf}\{F-\lambda L\}=-\infty$ for $|\lambda|>1$. The correct result will be derived in Section 3.

3.Equi-coercivity of a sequence of functionals

3.1 Preliminaries. Throughout this section (X,τ) satisfy (2.1).

Let F^ϵ be a sequence of functionals mapping X into $\overline{\mathbb{R}}$.

Definition 2. *$(F^\epsilon)_{\epsilon>0}$ $\tau-\Gamma$ converges to F iff :*

i) For every u in X, and every sequence u^ϵ in X τ -converging to u,

$$F(u)\leq \liminf_{\epsilon\to 0} F^\epsilon(u^\epsilon). \tag{3.1}$$

ii) For every u in X, there exists a sequence u^ϵ in X τ-converging to u such that:

$$F(u)\geq \limsup_{\epsilon\to 0} F^\epsilon(u^\epsilon). \tag{3.2}$$

It is important to note that F is τ-l.s.c. , even if the F^ϵ's are not. For any functional $F:\ X\to\overline{\mathbb{R}}$ its relaxed functional $\mathrm{cl}-F$ is defined as:

$$\mathrm{cl}-F=\mathrm{Sup}\ \{G,\ G(u)\leq F(u) \text{ for every } u \text{ in } X,\ G \text{ is } \tau \text{ l.s.c.}\} \tag{3.3}$$

Remark 1. An interesting property of Γ-convergence is that F^ϵ and $\mathrm{cl}-F^\epsilon$ have the same Γ-limit. Moreover $\mathrm{cl}-F$ is the Γ-limit of the sequence $F^\epsilon=F$.

Definition 3. *The family of functionals $(F^\epsilon)_{\epsilon>0}$ is said to be equi-coercive iff:*

$$(F^\epsilon(u^\epsilon))_{\epsilon>0} \text{ bounded in } \mathbb{R}\Rightarrow (u^\epsilon)_{\epsilon>0} \text{ bounded in } (X,\|.\|). \tag{3.4}$$

Proposition 2. *Let $(F^\epsilon)_{\epsilon>0}$ be a sequence of equi-coercive functionals $\tau-\Gamma$-converging to F. Then F is coercive.*

Proof of proposition 2. Assume that F is not coercive, i.e.:

$$\exists\ u_n\in X,\ \exists\ M\in\mathbb{R},\quad \lim_{n\to+\infty}\ \|u_n\|\ =+\infty,\quad F(u_n)\leq M.$$

It follows from point ii) of the definition of Γ-convergence that, for every n, there exists a sequence $(u_n^\epsilon)_{\epsilon>0}$ such that:

$$\tau-\lim_{\epsilon\to 0} u_n^\epsilon=u_n,\ \ \limsup_{\epsilon\to 0} F^\epsilon(u_n^\epsilon)\leq F(u_n)\leq M.$$

Using the τ-lower semi-continuity of the norm (which is a consequence of assumption (2.1)), we obtain:

$$\liminf_{\epsilon\to 0}\ \|u_n^\epsilon\|\geq\|u_n\|.$$

Hence the doubly indexed sequence a_n^ϵ :

$$a_n^\epsilon=\mathrm{Sup}\left\{\frac{1}{\|u_n^\epsilon\|},\quad M-F^\epsilon(u_n^\epsilon)\right\},$$

satisfies:

$$\limsup_{n\to+\infty}\ \limsup_{\epsilon\to 0} a_n^\epsilon\leq 0.$$

By the diagonalization lemma (ATTOUCH [1] p.33), we can choose a sequence $(n_\epsilon)_{\epsilon>0}$ such that :

$$\lim_{\epsilon\to 0}\ n_\epsilon=+\infty,\quad \limsup_{\epsilon\to 0} a_{n_\epsilon}^\epsilon\leq 0.$$

Define $v^\epsilon=u_{n_\epsilon}^\epsilon$, then:

$$\lim_{\epsilon\to 0}\ \|v^\epsilon\|=+\infty,\quad \limsup_{\epsilon\to 0} F^\epsilon(v^\epsilon)\leq M,$$

which contradicts the equi-coercivity of $(F^\epsilon)_{\epsilon>0}$. Q.E.D.

A classical result allows to pass to the limit in the sequence $(\mathrm{Inf}(F^\epsilon))_{\epsilon>0}$. More specifically (ATTOUCH [1]):

Proposition 3. *Let F^ϵ be a sequence of equi-coercive functionals on X, such that $(F^\epsilon)_{\epsilon>0}$ Γ-converges to F. Then:*

$$\lim_{\epsilon\to 0}\left(\mathrm{Inf}_{u\in X}\ F^\epsilon(u)\right)=\mathrm{Min}_{u\in X}\ F(u). \tag{3.5}$$

Moreover if u^ϵ is an approximate minimizer of F^ϵ:

$$F^\epsilon(u^\epsilon)\leq \mathrm{Inf}F^\epsilon+\epsilon,$$

then any cluster point u of $(u^\epsilon)_{\epsilon>0}$ for the topology τ is a minimizer of F:

$$u\in \mathrm{Argmin}(F).$$

Remark 2. If moreover F is proper, and since it is coercive (see Proposition 2), it results from Proposition 1 iii) that $\mathrm{Min}(F)$ is finite and consequently that F^ϵ is uniformly bounded from below.

3.2 The main result.

The main result of this section is:

Theorem 2. *Let $(F^\epsilon)_{\epsilon>0}$ be a sequence of proper, convex, equi-coercive functionals on X, $\tau-\Gamma$-convergent to F. Assume that (2.1) holds and that F is proper. Then i) and ii) are equivalent:*

i) $(F^\epsilon - \lambda L)_{\epsilon>0}$ *is an equi-coercive sequence of functionals,*

ii) $\underline{\lambda} < \lambda < \overline{\lambda}$ *where:*

$$\left.\begin{aligned} \overline{\lambda} &= \operatorname{Min}\{F_\infty(u),\ u\in X,\ L(u)=1\},\\ \underline{\lambda} &= -\operatorname{Min}\{F_\infty(u),\ u\in X,\ L(u)=-1\}\end{aligned}\right\} \tag{3.6}$$

An interesting consequence of Theorem 2 is obtained when F^ϵ is taken to be constant and equal to F. Since the F^ϵ's were not assumed to be τ-l.s.c. in Theorem 2 we obtain a useful generalization of Theorem 1:

Corollary 1. *Let F be a proper, convex and coercive functional on X. Then i) and ii) are equivalent:*

i)$F-\lambda L$ is coercive,

ii) $\underline{\lambda} < \lambda < \overline{\lambda}$ where:

$$\left.\begin{aligned} \overline{\lambda} &= \operatorname{Min}\{(\mathrm{cl}-F)_\infty(u),\ u\in X,\ L(u)=1\},\\ \underline{\lambda} &= -\operatorname{Min}\{(\mathrm{cl}-F)_\infty(u),\ u\in X,\ L(u)=-1\}\end{aligned}\right\}.$$

Proof of Theorem 2 : Preliminaries.

We can assume without loss of generality that $F^\epsilon(0)=0$. Indeed if this assumption is not met, according to the fact that F is proper, there exists u_0 in X where F is finite. By point ii) of the definition of Γ-convergence, there exists a sequence (u_0^ϵ) τ-converging to u and such that $F^\epsilon(u_0^\epsilon)$ converges to $F(u_0)$. Then set:

$$G^\epsilon(u) = F^\epsilon(u+u_0^\epsilon) - F^\epsilon(u_0^\epsilon),$$

and note that $G^\epsilon(0)=0$. It is straightforward to check that G^ϵ Γ-converges to G :

$$G(u) = F(u+u_0) - F(u_0), \text{ with } G(0)=0.$$

Moreover it is easily shown that the equi-coercivity of $(G^\epsilon)_{\epsilon>0}$ is equivalent to that of $(F^\epsilon)_{\epsilon>0}$ and that $G_\infty = F_\infty$. Hence it is sufficient to prove theorem 2 for the sequence $(G^\epsilon)_{\epsilon>0}$.

For simplicity we shall restrict our attention to positive λ's, and consider only the inequality $\lambda < \overline{\lambda}$.

Set $F_\lambda^\epsilon = F^\epsilon - \lambda L$ and note that since L is τ-continuous, $(F_\lambda^\epsilon)_{\epsilon>0}$ $\tau-\Gamma$-converges to $F_\lambda = F - \lambda L$. Note also that, since L is linear:

$$(F_\lambda)_\infty = (F_\infty)_\lambda = F_\infty - \lambda L. \tag{3.7}$$

First step. Proposition 3 applied to F_λ^ϵ proves that F_λ is coercive. Therefore, applying proposition 1 point iv) to F_λ, and after due account of (3.7), we obtain:

$$F_\infty(u) - \lambda L(u) > 0 \quad \forall u \in X - \{0\}, \tag{3.8}$$

Since $\overline{\lambda}$ is a minimum (and not only an infimum), it is reached by $\overline{u}$ such that:

$$F(\overline{u}) = \overline{\lambda}, \quad L(\overline{u}) = 1.$$

With this specific choice of u in (3.8) we obtain the desired inequality $\lambda < \overline{\lambda}$.

Second step. Let us prove that ii) implies i). Assume that

$$\lambda < \overline{\lambda}, \tag{3.9}$$

and that $(F_\lambda^\epsilon)_{\epsilon>0}$ is not equi-coercive, i.e. :

$$u^\epsilon \in X,\ \exists M \in \mathbb{R},\ \lim_{\epsilon\to 0} \| u^\epsilon \| = +\infty, \quad F_\lambda^\epsilon(u^\epsilon) \le M, \text{ i.e. :} \tag{3.10}$$

$$F^\epsilon(u^\epsilon) \le M + \lambda L(u^\epsilon). \tag{3.11}$$

Note that

$$\lim_{\epsilon\to 0} L(u^\epsilon) = +\infty,$$

otherwise (3.11) and the equi-coercivity of $(F^\epsilon)_{\epsilon>0}$ would prove that $(u^\epsilon)_{\epsilon>0}$ is a bounded sequence. Set :

$$v^\epsilon = \frac{u^\epsilon}{\| u^\epsilon \|}, \quad t^\epsilon = \| u^\epsilon \| .$$

Since $(v^\epsilon)_{\epsilon>0}$ is a sequence of norm 1, it is relatively τ-compact (assumption (2.1)) and it contains a subsequence, still denoted (v^ϵ) τ-converging to an element v of X. It follows from (3.11) that:

$$F^\epsilon(t^\epsilon v^\epsilon) \le M + \lambda t^\epsilon L(v^\epsilon). \tag{3.12}$$

Let t be a fixed positive scalar, then for ϵ small enough t is smaller than t^ϵ, which tends to $+\infty$, and by convexity of F^ϵ (recall that $F^\epsilon(0) = 0$):

$$\frac{F^\epsilon(tv^\epsilon)}{t} \leq \frac{F^\epsilon(t^\epsilon v^\epsilon)}{t^\epsilon} \leq \frac{M}{t^\epsilon} + \lambda L(v^\epsilon).$$

By point i) of the definition of Γ-convergence:

$$\frac{F(tv)}{t} \leq \liminf_{\epsilon \to 0} \frac{F^\epsilon(tv^\epsilon)}{t} \leq \lambda L(v). \tag{3.13}$$

Take the supremum of the left hand side of (3.13) on positive t to obtain:

$$F_\infty(v) \leq \lambda L(v). \tag{3.14}$$

F_∞ is positive (Proposition 1 point iv)), and it follows from (3.14) that $L(v)$ is positive (recall that we assumed $\lambda > 0$). If $L(v)$ does not vanish we set:

$$w = \frac{v}{L(v)}, \text{ i.e. } L(w) = 1,$$

and since F_∞ is positively homogeneous of degree one we derive from (3.14):

$$\overline{\lambda} \leq F_\infty(w) \leq \lambda,$$

which contradicts assumption (3.9). Therefore $L(v) = 0$, and by (3.14) $F_\infty(v) = 0$. It follows from Proposition 1 iv) that $v = 0$. Coming back to inequality (3.10) we obtain after due use of a previous remark ($\lim_{\epsilon \to 0} L(u^\epsilon) = +\infty$), associated with the convexity of F^ϵ :

$$F^\epsilon\left(\frac{u^\epsilon}{L(u^\epsilon)}\right) \leq \frac{F(u^\epsilon)}{L(u^\epsilon)} \leq \frac{M}{L(u^\epsilon)} + \lambda.$$

By equi-coercivity of $(F^\epsilon)_{\epsilon>0}$ we deduce that :

$$\frac{u^\epsilon}{L(u^\epsilon)} \text{ is a bounded sequence in } X,$$

and therefore that (since $u^\epsilon = t^\epsilon v^\epsilon$):

$$\frac{v^\epsilon}{L(v^\epsilon)} \text{ is a bounded sequence in } X. \tag{3.15}$$

(3.15) is in contradiction with:

$$\| v^\epsilon \| = 1 \text{ and } \lim_{\epsilon \to 0} L(v^\epsilon) = L(v) = 0. \qquad Q.E.D.$$

3.2 Examples.

In the forthcoming examples we shall use the following functional spaces:

$$BV(\Omega) = \left\{ f \in L^1(\Omega),\ \frac{\partial f}{\partial x_i} \in M^1(\Omega),\ i = 1, N \right\}, \tag{3.16}$$

$$BD(\Omega) = \left\{ u = (u_i)_{i=1,N},\ u_i \in L^1(\Omega),\ \epsilon_{ij}(u) \in M^1(\Omega) \right\}, \tag{3.17}$$

$$LD(\Omega) = \left\{ u = (u_i)_{i=1,N},\ u_i \in L^1(\Omega),\ \epsilon_{ij}(u) \in L^1(\Omega) \right\},$$

where $M^1(\Omega)$ stands for the space of bounded measures on Ω. Classical results establish that $BD(\Omega)$ (respectively $BV(\Omega)$) has the following properties:

i) $BD(\Omega)$, respectively $BV(\Omega)$, is the dual of a Banach space, and therefore, can be endowed with a weak * topology for which bounded sets are relatively compact sets. However $BD(\Omega)$, respectively $BV(\Omega)$, is not a reflexive space.

ii) There exists a trace application mapping $BD(\Omega)$ onto $L^1(\partial\Omega)^N$, respectively mapping $BV(\Omega)$ onto $L^1(\partial\Omega)$, continuous for the strong topologies of these two spaces. However this trace application is not continuous for the weak * topologies of these spaces.

iii) $BD(\Omega)$, respectively $BV(\Omega)$, is continuously embedded into $L^p(\Omega)^N$, respectively $L^p(\Omega)$, for $1 \leq p \leq N/(N-1)$, with compact embedding for $p < N/(N-1)$.

We shall also make use of functions of a measure ([12], [7]). Let $h : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ be a Borelian function, positively homogeneous of degree 1, *i.e.* such that:

$$h(x, \lambda E) = \lambda h(x, E) \quad \forall \lambda > 0, \quad \forall x \in \Omega, \tag{3.18}$$

then for every Borelian measure $\mu : \Omega \to \mathbb{R}^d$ and every positive measure θ such that μ is absolutely continuous with respect to θ,

$$\mu = \frac{d\mu}{d\theta}\theta \quad \text{where} \quad \frac{d\mu}{d\theta} \in L^1(\Omega, \theta),$$

the integral:

$$\int_\Omega h(x, \frac{d\mu}{d\theta}) d\theta \tag{3.19}$$

does not depend on θ. The common value of all these integrals is denoted:

$$\int_\Omega h(x, \mu). \tag{3.20}$$

1. Plasticity. The equilibrium problem of an elasto-plastic body obeying Hencky's law of Plasticity and submitted to a loading proportional to a scalar parameter λ, can be formulated in term of displacements as a minimization problem:

$$\operatorname*{Inf}_{u = u_0 \text{ on } \Gamma_0} \left\{ \int_\Omega j^\epsilon(x, e(u))\, dx - \lambda L(u) \right\}. \tag{3.21}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $u:\ \Omega \to \mathbb{R}^N$ is a vector valued field (rate of displacement), $e(u)$ is its deformation tensor, Γ_0 is the subset of the boundary $\partial\Omega$ where the body is fixed. $j^\epsilon(x, E) = j(\frac{x}{\epsilon}, E)$ is ϵ-periodic with respect to the variable x:

$$j^\epsilon(x + \epsilon T, E) = j^\epsilon(x, E) \quad \forall T \in \mathbb{Z}^N. \tag{3.22}$$

j is assumed to be convex with respect to E and to satisfy:

$$k_0\left(|E^D| + |trE|^2 - 1\right) \le j(y, E) \le k_1\left(|E^D| + |trE|^2 + 1\right). \tag{3.23}$$

$L(u)$ is the linear form:

$$L(u) = \int_\Omega f_0 u\, dx + \int_{\Gamma_1} g_0 u\, ds,$$

where $f_0 \in (L^N(\Omega))^N$, $g_0 \in (L^\infty(\Gamma_1))^N$ and $\Gamma_1 = \partial\Omega - \Gamma_0$. The function j, which plays the role of a non quadratic energy, can be further specified when the material obeys Hooke's law in the elastic regime with Lamé coefficients λ and μ, and Von Mises criterion with shear strength k in the plastic regime (TEMAM [15] p 76)

$$j(y, E) = \varphi(y, Tr(E)) + \psi(y, E^D), \tag{3.24}$$

where

$$\psi(y, E) = \begin{cases} \mu(y)|E|^2 \text{ if } |E| \le \dfrac{k(y)}{\sqrt{2}\mu(y)} \\ \sqrt{2}k(y)|E| - \dfrac{k(y)^2}{2\mu(y)} \text{ otherwise} \end{cases},$$

$$\varphi(y, T) = \frac{1}{2}\left(\frac{3\lambda(y) + 2\mu(y)}{3}\right)(T)^2,$$

and where the following notations have been used

$$TrE = \sum_{i=1}^{3} E_{ii}, \quad E^D_{ij} = E_{ij} - \frac{TrE}{3}\,\delta_{ij}\ , |E|^2 = \sum_{i,j=1}^{3} E^2_{ij} \quad (\text{in } \mathbb{R}^3).$$

Let $u_0 \in (H^{1/2}(\Gamma_0))^N$, and define on $BD(\Omega)$:

$$J^\epsilon_{u_0}(u) = \begin{cases} \int_\Omega j^\epsilon(e(u))\, dx \text{ if } u \in LD(\Omega) \text{ and } u = u_0 \text{ on } \Gamma_0, \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{3.25}$$

Let $j^{hom}(E)$ be the homogenized density of energy:

$$j^{hom}(E) = \operatorname*{Inf}_{w \in \mathbf{H}^1_{per}(]0,1[^N)} \left\{ \frac{1}{|Y|} \int_Y j(y, E + e(w))dy \right\}.$$

$\mathbf{H}^1_{per}(]0,1[^N)$ stands for the periodic vector fields with components in $H^1(]0,1[^N)$, and define for every u in $BD(\Omega)$:

$$J^{hom}_{u_0}(u) = \int_\Omega j^{hom}(e(u)) + \int_{\Gamma_0} j^{hom}_\infty((u_0 - u) \otimes_s n)\, ds. \tag{3.26}$$

The integral over Ω of $j^{hom}(e(u))$, when $e(u)$ is a measure, is understood in the following sense:

$$\int_\Omega j^{hom}(e(u)) = \int_\Omega j^{hom}(e_a(u))\, dx + \int_\Omega j^{hom}_\infty(e_s(u)),$$

where $e_a(u)$ and $e_s(u)$ denote respectively the absolutely continuous part and the singular part of $e(u)$ with respect to the Lebesgue measure dx, and where the integral of $j^{hom}_\infty(e_s(u))$ over Ω is understood in the sense of functions of a measure (see (3.20)).

Theorem 2 may be applied with the following choice:

$$X = BD(\Omega),\ \tau = \text{strong topology of } L^p(\Omega)^N\ (p < N/(N-1)).$$

As previously stated by BOUCHITTE [4] (extended in [11]), $J^\epsilon_{u_0}$ τ-Γ-converges to $J^{hom}_{u_0}$. Moreover it is easy to check that $J^\epsilon_{u_0}$ is equi-coercive on $BD(\Omega)$ provided that:

$$\text{mes}(\Gamma_0) > 0,\ 0 < k_0 \le k(y) \le k_1 < +\infty \quad \forall y \in]0,1[^N. \tag{3.27}$$

It remains to check that L is τ-continuous and this turns out to be true only if $g_0 = 0$ or if $\Gamma_1 = \emptyset$, i.e. if the body is not loaded on its boundary. In this case we recover a result already derived in BOUCHITTE [4] by different means:

Proposition 4 : *With the previous notations assume that $g_0 = 0$ or that $\Gamma_1 = \emptyset$. Then the sequence of functionals $(J^\epsilon_{u_0} - \lambda L)_{\epsilon > 0}$ is equi-coercive in $BD(\Omega)$ if and only if:*

$$\underline{\lambda} < \lambda < \overline{\lambda},$$

where:

$$\left.\begin{aligned}\overline{\lambda} &= \operatorname{Inf}\left\{J_{\infty}^{hom}(u),\ L(u)=1\right\},\\ \underline{\lambda} &= -\operatorname{Inf}\left\{J_{\infty}^{hom}(u),\ L(u)=-1\right\},\end{aligned}\right\} \qquad (3.28)$$

and

$$J_{\infty}^{hom}(u) = \int_{\Omega} j_{\infty}^{hom}(e(u)) + \int_{\Gamma_0} j_{\infty}^{hom}(-u \otimes_s n)\, ds \text{ if } u \in BD(\Omega).$$

Remark 3: The homogeneous case considered in TEMAM [15] can be recovered from Proposition 4. For this purpose consider $j^{\epsilon}(x,E) = j(E) = j^{hom}(E)$ independent of ϵ. Then (3.28) defines the classical limit load problem.

When $\Gamma_0 \neq \emptyset$ and $g_0 \neq 0$, L is no more τ-continuous and Proposition 4 does not hold (see a counterexample in [6]). This difficulty can be overcome by considering separately the contribution of u inside Ω and on the boundary $\partial\Omega$. The technical details will not be given here, since the next example is similar in its principle. The interested reader is referred to [5] [6].

2. Capillarity problem (second part).

In the capillarity problem considered in section 2.2 recall that:

$$X = W^{1,1}(\Omega),\ \tau = \text{strong topology of } L^p(\Omega),\ L(u) = \int_{\partial\Omega} u ds.$$

As noted previously, the linear term L is not τ-continuous. Therefore the contributions of u inside Ω and on $\partial\Omega$ are considered separately. More specifically set:

$$X = BV(\Omega) \times M^1(\partial\Omega),$$

$$\tau = \text{weak topology of } L^{N/(N-1)}(\Omega) \times \text{weak } * \text{ topology of } M^1(\partial\Omega),$$

$$\Phi(u,\mu) = \begin{cases} F(u) \text{ if } u \in BV(\Omega) \text{ and } \mu = uds \text{ on } \partial\Omega, \\ +\infty \quad \text{otherwise} \end{cases}$$

$$\hat{L}(u,\mu) = \int_{\partial\Omega} d\mu.$$

F is defined in (2.12). Note that Φ is proper, convex and τ-l.s.c. on X and that $\hat{L}$ is now τ-continuous. Moreover it is easy to check that:

i) $\displaystyle \operatorname*{Inf}_{u \in W^{1,1}(\Omega)} \{F(u) - \lambda L(u)\} = \operatorname*{Inf}_{(u,\mu)\in X} \left\{\Phi(u,\mu) - \lambda \hat{L}(u,\mu)\right\}$

ii) $F - \lambda L$ coercive on $W^{1,1}(\Omega) \Leftrightarrow \Phi - \lambda\hat{L}$ coercive on X.

By virtue of Corollary 1 :

$$F - \lambda L \text{ coercive} \Leftrightarrow |\lambda| < \operatorname{Inf}\left\{(\bar{\Phi})_{\infty}(u),\quad L(u)=1\right\}.$$

The difficulty relies in the computation of the τ-closure of Φ, which is performed by duality. The conjugate function of Φ on $L^N(\Omega) \times C^0(\partial\Omega)$ reads as:

$$\Phi^*(f,\varphi) = \operatorname*{Inf}_{p_0,p_1} \left\{\int_{\Omega}\left(-\sqrt{1-|p_1|^2} + \frac{1}{2k}|p_0|^2\right) dx\right\},$$

where the infimum is taken over (p_0,p_1) such that:

$$p_0 - \operatorname{div}(p_1) = f \text{ in } \Omega,\ |p_1(x)| \leq 1 \text{ a.e. in } \Omega,\ p_1.n = \varphi \text{ on } \partial\Omega.$$

Since Φ is convex and proper its τ-closure is equal to Φ^{**}. Using the same technical arguments than in BOUCHITTE & SUQUET [6], we obtain:

$$\text{cl} - \Phi(u,\mu) = \int_{\Omega}\sqrt{1+|\nabla_a u|^2}dx + \int_{\Omega}|\nabla_s u| + \frac{k}{2}\int_{\Omega}|u|^2 dx + \int_{\partial\Omega} h(x,\mu),$$

where $h(x,.) = \left(\mathbb{I}_{C(x)}\right)^*$. $x \to C(x)$ is a l.s.c. multi-function and $C(x)$ is the closure of the set

$$\{z \in \mathbb{R}, \exists\, \varphi \in C^0(\partial\Omega) : \varphi(x) = z,$$

$$\exists\, \theta \in L^{\infty}(\Omega) :\ |\theta(x)| \leq 1 \text{ a.e. in } \Omega,\ \operatorname{div}(\theta) \in L^N(\Omega),\ \theta.n = \varphi \text{ on } \partial\Omega\}.$$

By symmetry there exists a l.s.c. function $a(x)$ such that $C(x) = \{|z| \leq a(x)\}$ and therefore $h(x,s) = a(x)|s|$. The computation of $(\text{cl} - \Phi)_{\infty}$ is straightforward:

$$(\text{cl}_{\Phi})_{\infty} = \begin{cases} \int_{\partial\Omega} a(x)\, d|\mu| & \text{if } u = 0 \text{ in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence:

$$\overline{\lambda} = \operatorname*{Min}_{x \in \partial\Omega} a(x).$$

When the boundary $\partial\Omega$ is C^1 it is easy to check that $a(x) = 1$ and $\overline{\lambda} = 1$. However when the boundary $\partial\Omega$ is only piecewise C^1, i.e. exhibits corners, a lengthy computation leads to:

$$a(x) = \left(\frac{1 + n_-(x).n_+(x)}{2}\right)^{1/2},$$

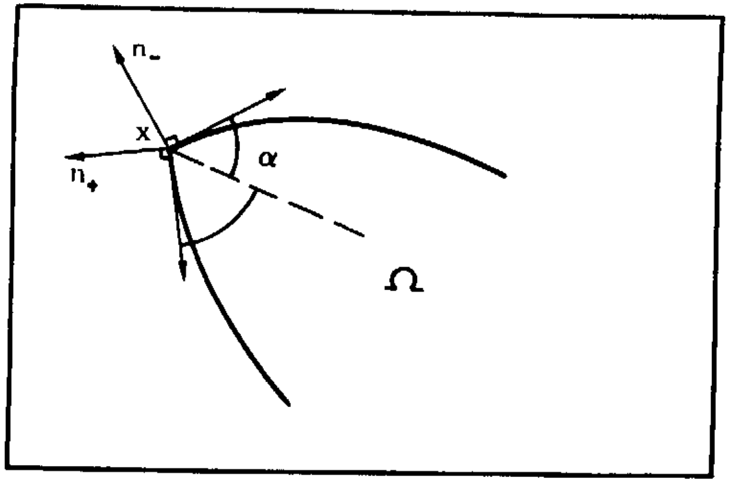

Figure 1

where $n_-(x) = n_+(x)$ at points x where the outer normal vector is uniquely defined, and takes two different values at corners. The critical value of λ is:

$$\overline{\lambda} = \underset{x \in \partial\Omega}{\text{Min}} \ \{\sin(\alpha(x))\},$$

which is a result already derived by MASSARI & MIRANDA [13] by different means.

4. Γ-Convergence of recession functions

4.1 Preliminaries

Theorems 1 and 2 put forth the role played by recession functions in variational problems. A natural question arises:

Consider a sequence of functionals $(F^\epsilon)_{\epsilon>0}$ which Γ-converges to a functional F, does the sequence of recession functions $(F^\epsilon_\infty)_{\epsilon>0}$ (when they can be defined) Γ-converge to F_∞?

A weaker, but related question, is:

Let L be a continuous linear form, does the sequence of infimas

$$\overline{\lambda}^\epsilon = \underset{u}{\text{Inf}} \ \{F^\epsilon_\infty(u),\ L(u) = 1\}, \tag{4.1}$$

converge to

$$\overline{\lambda} = \underset{u}{\text{Inf}} \ \{F_\infty(u),\ L(u) = 1\} \ \ ? \tag{4.2}$$

Without stronger assumptions the answer to these questions is negative as illustrated by the following example. Consider on $X = \mathbb{R}^N$ the sequence:

$$F^\epsilon(u) = \|u\| + \epsilon\|u\|^2,$$

which Γ-converges to $F(u) = \|u\|$. Note that $F^\epsilon_\infty = \mathbb{I}_{\{0\}}$ which Γ-converges to $\mathbb{I}_{\{0\}}$, but $F_\infty = F = \|.\|$. Consider the linear form $L(u) = (u^*, u)$ where $u^* \in \mathbb{R}^N$. Then:

$$\overline{\lambda}^\epsilon = \text{Inf}\,\{F^\epsilon_\infty(u),\quad L(u) = 1\} = +\infty,$$

but

$$\overline{\lambda} = \text{Inf}\,\{F_\infty(u),\quad L(u) = 1\} = \frac{1}{\|u^*\|}. \qquad . \quad (Q.E.D.)$$

However the answers to the previous questions are positive under additional hypotheses. Assume that:

$$X \text{ is compactly embedded in a reflexive Banach space } Y, \tag{4.3}$$

and that τ is the topology induced on X by the weak or strong topology of Y (so (2.1) holds). By convention we extend every functional defined on X to Y by giving the value $+\infty$ in Y-X.

Theorem 3 . *Let X satisfy (4.3), and let $(F^\epsilon)_{\epsilon>0}$ be a sequence of convex, proper and l.s.c. (in X strong) functionals mapping X into $\overline{\mathbb{R}}$ such that the following hypotheses are satisfied:*

i) $\text{Inf} F^\epsilon > -\infty$,

ii) $(F^\epsilon)_{\epsilon>0}$ *τ-Γ-converges to a proper functional F,*

iii) *for every sequence $(f^\epsilon)_{\epsilon>0}$ in Y^* such that $(F^\epsilon)^*(f^\epsilon) < +\infty$, then $(F)^*(f) < +\infty$ for any weak cluster point f in Y^* of $(f^\epsilon)_{\epsilon>0}$.*

Then:

$$F^\epsilon_\infty \ \tau - \Gamma - \text{converges to } F_\infty. \tag{4.4}$$

Comment: Theorem 3 does not require that the F^ϵ's are l.s.c. for the topology τ but only for the strong topology of X. This assumption is sufficient to define F^ϵ_∞ properly.

Proof of Theorem 3. Point i) of definition 2 is straightforwardly satisfied. Indeed, as in the proof of Theorem 2 we can assume without loss of generality that $F^\epsilon(0) =$

$F(0)=0$. Then for every v in X, for every sequence $(v^\epsilon)_{\epsilon>0}$ τ-converging to v and every $t>0$:

$$\liminf_{\epsilon\to 0} F^\epsilon_\infty(v^\epsilon) \geq \liminf_{\epsilon\to 0} \frac{F^\epsilon(tv^\epsilon)}{t} \geq \frac{F(tv)}{t}. \tag{4.5}$$

Then (3.1) is deduced by taking the supremum over t in the right hand side of (4.5).

Let us now prove (3.2). According to Remark 1 in section 3 we can assume that F^ϵ, hence F^ϵ_∞, is τ-l.s.c. on Y, since it can be replaced by its τ-closure with no modification of its Γ-limit. We use a duality argument (duality Y, Y^*) in order to establish point ii) of the definition of Γ-limit (Definition 2). Following ATTOUCH ([1] p.271) the desired assertion:

$$\forall u \in Y,\ \exists u^\epsilon \xrightarrow{\tau} u,\ F_\infty(u) \geq \limsup_{\epsilon\to 0} F^\epsilon_\infty(u^\epsilon), \tag{4.6}$$

is a consequence of the two assertions:

$$\exists f_0 \in Y^*, \quad (F^\epsilon_\infty)^*(f_0) < +\infty, \tag{4.7}$$

$$\forall f^\epsilon \to f \text{ in } Y^* \text{ weak}, \quad (F_\infty)^*(f) \leq \liminf_{\epsilon\to 0} (F^\epsilon_\infty)^*(f^\epsilon). \tag{4.8}$$

To check (4.7) note that :

$$(F^\epsilon_\infty)^* = \mathbf{I}_{\mathrm{cl}-\mathrm{dom}\,(F^\epsilon)^*},$$

i.e.

$$(F^\epsilon_\infty)^* = 0 \quad \text{or} \quad +\infty.$$

But 0 belongs to $\mathrm{dom}\,(F^\epsilon)^*$ since $\mathrm{Inf} F^\epsilon > -\infty$ (assumption i)). Therefore

$$(F^\epsilon_\infty)^*(0) = 0.$$

To check (4.8) assume that the right hand side of (4.8) is finite (otherwise there is nothing to prove). Then there exists a subsequence, still denoted f^ϵ, such that:

$$f^\epsilon \in \mathrm{cl}-\mathrm{dom}\,(F^\epsilon_\infty)^* \text{ and } f^\epsilon \to f \text{ in } Y^* \text{ weak},$$

and a sequence g^ϵ approximating f^ϵ in Y^* weak such that

$$g^\epsilon \in \mathrm{dom}\,(F^\epsilon_\infty)^*, \quad g^\epsilon \to f \text{ in } Y^* \text{ weak}.$$

We deduce from assumption iii) applied to g^ϵ, that f (weak cluster point of g^ϵ) belongs to $\mathrm{dom}(F^*)$ i.e. $(F_\infty)^*(f) = 0$. Q.E.D.

Corollary 2. *Let L be a linear continuous form on Y, let $(F^\epsilon)_{\epsilon>0}$ be as in Theorem 3 and equi-coercive. Then :*

$$\lim_{\epsilon\to 0} \operatorname*{Inf}_u \{F^\epsilon_\infty(u), \quad L(u)=1\} = \operatorname*{Inf}_u \{F_\infty(u), \quad L(u)=1\}. \tag{4.9}$$

Proof of Corollary 2. As previously we can assume with no loss of generality that $F^\epsilon(0)=F(0)=0$. Then $F^\epsilon_\infty \geq F^\epsilon$ and $(F^\epsilon_\infty)_{\epsilon>0}$ is equi-coercive as soon as $(F^\epsilon)_{\epsilon>0}$ is equi-coercive. In order to apply Theorem 3 it is sufficient to prove that $F^\epsilon_\infty + \mathbf{I}_{\{L(u)=1\}}$ τ-Γ-converges to $F_\infty + \mathbf{I}_{\{L(u)=1\}}$. This is a straightforward consequence of Theorem 3 and of the positive homogeneity of degree one of F^ϵ_∞ and F_∞.

4.2 Example: Plasticity.

Consider again the homogenization problem in Plasticity as described in section 3.2 with the simplifying assumption $\Gamma_0 = \partial\Omega$. We apply Theorem 3 and Corollary 2 with:

$$X = BD(\Omega), \quad Y = L^p(\Omega)^N,\ 1 < p < N/(N-1),$$

$$L(u) = \int_\Omega f_0 u \, dx, \quad f_0 \in L^{p'}(\Omega)^N, \quad F^\epsilon = J^\epsilon_{u_0} \text{ (see (3.25))}.$$

We emphasize that F^ϵ is l.s.c. for the strong topology of X but not τ-l.s.c. With the notations of section 3.2

$$F^\epsilon_\infty(u) = \begin{cases} \displaystyle\int_\Omega \psi_\infty\left(\frac{x}{\epsilon}, e(u)\right) dx \text{ if } u \in LD(\Omega),\ \mathrm{div}(u)=0,\ u=0 \text{ on } \Gamma_0, \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{4.10}$$

where

$$\psi_\infty(y,E) = \sqrt{2}k(y)\|E\|.$$

The "limit load problem" reads as (TEMAM [15]):

$$\overline{\lambda}^\epsilon = \operatorname*{Inf}_{u\in LD(\Omega)} \left\{ \int_\Omega \psi_\infty\left(\frac{x}{\epsilon}, e(u)\right) dx,\ L(u)=1,\ \mathrm{div}(u)=0,\ u=0 \text{ on } \Gamma_0 \right\}.$$

Define on $BD(\Omega)$:

$$J^{hom}_\infty(u) = \begin{cases} \displaystyle\int_\Omega \psi^{hom}_\infty(e(u)) + \int_{\Gamma_0} \psi^{hom}_\infty(-u_\tau \otimes_s n) ds \\ \quad \text{if } \mathrm{div}(u)=0,\ u.n=0 \text{ on } \Gamma_0 \\ +\infty \qquad \text{otherwise.} \end{cases},$$

where $u_\tau = u - (u.n)n$ is the tangential component of u on Γ_0, and:

$$\psi_\infty^{hom}(E) = \begin{cases} \underset{\varphi \in \mathbf{H}^1_{per}(Y)}{\operatorname{Inf}} \left\{ \frac{1}{|Y|} \int_Y \psi_\infty(y, E + e(\varphi))dy \right\} & \text{if } Tr(E) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Set :

$$\overline{\lambda} = \underset{u \in LD(\Omega)}{\operatorname{Inf}} \{ J_\infty^{hom},\ L(u) = 1, \}$$

Proposition 5. *With the above notations:*

i) $(J^\epsilon_{u_0})_\infty$ *Γ-converges to* J_∞^{hom},

ii) $\lim_{\epsilon \to 0} \overline{\lambda}^\epsilon = \overline{\lambda}$.

Proof of Proposition 5. Proposition 5 will result from Theorem 3 and Corollary 2, provided that all the assumptions are met. To check that point i) of Theorem 3 is met, note that $F^\epsilon = J^\epsilon_{u_0}$ is positive and proper. Therefore:

$$0 \le \operatorname{Inf} F^\epsilon < +\infty.$$

To check point ii) recall that $J^\epsilon_{u_0}$ Γ-converges to $J^{hom}_{u_0}$ (3.26) which is proper. It remains to check point iii).

Let $(f^\epsilon)_{\epsilon>0}$ be a sequence in $L^{p'}(\Omega)^N$ weakly converging to f and such that

$$(F^\epsilon)^*(f^\epsilon) < +\infty.$$

To simplify notations we shall assume from now on that $u_0 = 0$. A standard computation (TEMAM [15] for similar computations) yields:

$$(F^\epsilon)^*(f^\epsilon) = \underset{\sigma^D,\ p}{\operatorname{Inf}} \left\{ \int_\Omega \left(\psi^*\left(\frac{x}{\epsilon}, \sigma^D\right) + \varphi^*\left(\frac{x}{\epsilon}, p\right)\right) dx, \right\},$$

where the infimum is taken over $(\sigma^D,\ p)$ such that

$$\left.\begin{array}{c} \operatorname{grad}(p) - \operatorname{div}(\sigma^D) = f^\epsilon \\ \operatorname{Tr}(\sigma^D) = 0 \end{array}\right\}$$

and where

$$\psi^*(y, \sigma^D) = \frac{1}{4\mu(y)} \|\sigma^D\|^2 + \mathbf{I}_{\{\|\sigma^D\| \le \sqrt{2}k(y)\}},$$

and

$$\varphi^*(y,p) = \frac{1}{2} \frac{1}{3\lambda(y) + 2\mu(y)} p^2.$$

Using assumption (3.27) which ensures that ψ^* is bounded on its domain, and using the fact that $(F^\epsilon)^*(f^\epsilon) < +\infty$, we deduce that there exists a sequence $(\sigma^D_\epsilon,\ p_\epsilon)$ such that:

$$\left.\begin{array}{c} \operatorname{grad}(p_\epsilon) - \operatorname{div}(\sigma^D_\epsilon) = f^\epsilon, \\ \|\sigma^D_\epsilon\| \le C, \\ (F^\epsilon)^*(f^\epsilon) \le C\left(1 + \|p_\epsilon\|_{L^2(\Omega)/\mathbb{R}}\right) \end{array}\right\} \quad (4.11)$$

$(f^\epsilon)_{\epsilon>0}$ is bounded in $L^{p'}(\Omega)^N$ and σ^D_ϵ in $L^\infty(\Omega)^N$. Hence $\operatorname{grad}(p_\epsilon)$ is bounded in $H^{-1}(\Omega)$. According to DENY and LIONS ([15] proposition 1.2 p 16) this implies that $(p^\epsilon)_{\epsilon>0}$ is bounded in $L^2(\Omega)/\mathbb{R}$. Coming back to (4.11) we deduce that $(F^\epsilon)^*(f^\epsilon)$ is uniformly bounded and using the Γ-convergence of F^ϵ to F we deduce:

$$(F)^*(f) \le \liminf_{\epsilon \to 0} (F^\epsilon)^*(f^\epsilon) < +\infty,$$

which completes the proof of point iii) of Theorem 3. In order to apply Corollary 2, it is sufficient to check that F^ϵ is equi-coercive. This is a straightforward consequence of assumption (3.27).

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A CAPIETTO, J MAWHIN AND F ZANOLIN

Boundary value problems for forced superlinear second order ordinary differential equations

1 Introduction

This paper describes recent results obtained in the existence and multiplicity of solutions of nonlinear ordinary differential equations of the form

$$u''(t) + g(u(t)) = p(t, u(t), u'(t)),\ t \in [a, b], \tag{1}$$

satisfying boundary conditions of the Sturm-Liouville or periodic type at a and b, when $g : \mathbb{R} \to \mathbb{R}$ is continuous and *superlinear*, i.e.

$$\frac{g(u)}{u} \to +\infty \text{ as } |u| \to +\infty, \tag{2}$$

and $p : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a linear growth condition in the last two arguments. Problems of this type have been considered since the late fifties, among others, by Ehrmann [7], Morris [12], Fučik-Lovicar [8], Struwe [14] using shooting arguments and by Bahri-Berestycki [1],[2], Rabinowitz [13], Long [10] using critical point theory. The reader can consult [3] for more details and references. It may look surprising that one had to wait for the nineties to see the method of Leray-Schauder applied to such problems (see [3]). The reason can be found in the fact that the success of the Leray-Schauder method relies upon the obtention of a priori estimates for the possible solutions of a family of equations connecting (1) to a simpler problem for which the corresponding topological degree is not zero. A natural choice for (1) would be to consider the family of equations

$$u''(t) + g(u(t)) = \lambda p(t, u(t), u'(t)),\ \lambda \in [0, 1], \tag{3}$$

which reduces to (1) when $\lambda = 1$ and to the simple autonomous equation

$$u''(t) + g(u(t)) = 0, \tag{4}$$

when $\lambda = 0$. An elementary study of (4) under condition (2), based upon the first integral of energy, reveals that (4) will have infinitely many solutions with arbitrary large amplitudes, satisfying the boundary conditions, and the above mentioned