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Existence and Regularity of Solutions for Plasticity Problems

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ABSTRACT

This work examines the problem of the quasi-static evolution of perfectly plastic bodies.

In a first section a variational formulation is given in an adequate functional framework. The elements of the space of admissible velocity fields have a tensor of deformation which is composed of measures.

An existence theorem is stated. A second section deals with the regularity of the elements of this new space and it is shown that discontinuities (slip lines) might appear in the body. In the last section, a mathematical formalism is developed in order to consider the boundary as part of the body, and to explain the noncompliance of boundary conditions as a slip line phenomenon.

1. ELASTO-PLASTIC PROBLEMS

We are dealing in this work with the quasi-static evolution of perfectly plastic bodies. There are many publications concerning the stress problem (Moreau [1], Duvaut and Lions [2], and their schools), but only a few about the displacement problem [3,4]. The main difficulty is that, due to the noncoercivity of the plastic potential of dissipation, the classical functional Sobolev Spaces cannot be chosen as variational spaces for plasticity problems. Recently, the following variational framework has been proposed [5,6]: *"space of fields with bounded deformation"*:

$$BD(\Omega) = \{u \mid u = (u_i),\ u_i \in L^1(\Omega),\ \varepsilon_{ij}(u) \in M^1(\Omega),\ 1 \le i,j \le N\},$$

where $M^1(\Omega)$ is the space of bounded measures and N is the dimension of the physical space ($1 \le N \le 3$).

Our goal is to give an existence theorem for the problem of quasi-static evolution (Section 1), the velocity being in $BD(\Omega)$. The regularity of the velocity solutions and the boundary conditions are discussed in Sections 2 and 3, respectively.

1.1 Basic Equations

The total strain is the sum of an elastic part and of an anelastic part:

$$\left.\begin{aligned} \varepsilon_{ij} &= \varepsilon^e_{ij} + \varepsilon^{an}_{ij} \\ \varepsilon^e_{ij} &= A_{ijkh}\,\sigma_{kh} \end{aligned}\right\}, \tag{1.1}$$

where A and σ denote the elasticity coefficients and the stress field. With the chosen flow rule (Prandtl-Reuss) the plastic rate of deformation $\dot\varepsilon^{an}_{ij} = d\varepsilon^{an}_{ij}/dt$ must satisfy

$$(\dot\varepsilon^{an}_{ij}(x),\ \tau_{ij} - \sigma_{ij}(x)) \le 0\ (\forall\,\tau \in B(x));\ \sigma(x) \in B(x)\ , \tag{1.2}$$

where B(x) is the elasticity convex at point x.

It is convenient to introduce a regularized version of (1.2), namely a viscoplastic law obtained

by penalizing B(x):

$$\dot\varepsilon^{an}_{ij}(x) = \frac{1}{\mu}\,(\sigma_{ij}(x) - \Pi_{B(x)}\,\sigma_{ij}(x)) = (J'_\mu(\sigma))_{ij}\ . \tag{1.3}$$

Π_B denotes the projection on B in R^{N2}; μ is the viscosity. The conservation laws and the quasi-static assumption lead to the equilibrium equations:

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega \quad (f : \text{volumic loads})\ . \tag{1.4}$$

The boundary conditions are

$$u_i = U^d_i \ \text{on } \Gamma_U\ ;\ \sigma_{ij}n_j = F^d_i \quad \text{on } \partial\Omega - \Gamma_U\ . \tag{1.5}$$

The initial state is given: $\sigma(0) = \sigma_0$; $u(0) = u_0$.

The perfectly plastic problem consists in solving Eqs. (1.1), (1.2), (1.4), (1.5), while the visco-plastic problem is contained in Eqs. (1.1), (1.3)-(1.5).

Throughout the following, it is assumed that the space of statically admissible fields is not empty (i.e. Eqs. (1.4), (1.5) admit at least one solution), and that the data are regular.

1.2 Variational Formulations

The strong variational formulation of the viscoplastic problem is

Find $\sigma^\mu(t)$, $v^\mu(t)$ *satisfying*

$$\left.\begin{aligned} &A(\dot\sigma^\mu, \tau) + (J'_\mu(\sigma^\mu), \tau) = (\varepsilon(v^\mu), \tau) \quad \forall\tau \in H \\ &(D\sigma^\mu, w) + (f, w) = 0 \qquad \forall w \in V \end{aligned}\right\} \tag{1.6}$$

plus initial and boundary conditions.

We have set

$$v^\mu = \dot u^\mu$$

$$H = \{\tau \mid \tau = (\tau_{ij})\ ,\ \tau_{ij} = \tau_{ij}\ ,\ \tau_{ij} \in L^2(\Omega)\}$$

$$A(\sigma, \tau) = \int_\Omega A_{ijkh}(x)\,\sigma_{kh}(x)\,\tau_{ij}(x)\,dx$$

with the usual assumptions on the coefficients A_{ijkh}, the bilinear form A is symmetric, continuous and coercive on H.

$$D\tau = (\tau_{1j,j}, \ldots, \tau_{Nj,j})$$

$$H_{ad} = \{\tau \mid \tau \in H,\ D\tau \in (L^2(\Omega))^N,\ \tau_{ij}n_j = F^d_i \ \text{on } \Gamma_F\}$$

$$P = \{\tau \mid \tau \in H,\ \tau(x) \in B(x) \ \text{a.e. in } \Omega\}$$

$$V = (L^2(\Omega))^N\ .$$

In order to establish a variational formulation for the perfectly plastic problem, a first reaction would be to pass to the limit in (1.6) and to assert that the solution of (1.1), (1.2), (1.4), (1.5) satisfies the following *strong* variational formulation of the plasticity problem.

Find $\sigma(t) \in P$, $v(t)$ *such that*

$$\left.\begin{aligned} &A(\dot\sigma, \tau - \sigma) - (\varepsilon(v), \tau - \sigma) \ge 0 \ (\forall\tau \in P) \\ &(D\sigma, w) + f, w) = 0 \qquad \forall w \in V \end{aligned}\right\} \tag{1.7}$$

plus initial and boundary conditions.

This formulation is not the good one, since it is too strong and cannot be solved for two reasons: The first one is that τ belonging to P is not necessarily $L^1(\Omega, \varepsilon(v))$. It means that the duality $M^1(\Omega)$ (for the strain, $L^\infty(\Omega)$ (for the stresses) is not good enough; $M^1(\Omega) - C^\circ(\bar{\Omega})$ would be better, but except in particular cases [6] the stress field σ(or τ) is not in $C^\circ(\bar{\Omega})$.

The second reason is that v does not necessarily obey the boundary conditions. We shall return to this point in the third section.

For these two reasons, we must weaken the strong formulation (1.7) by using a formal Green formula:

$$(\varepsilon(v), \tau - \sigma) + (v, D\tau - D\sigma) = \int_{\Gamma_u} \dot{U}_i^d(\tau_{ij} - \sigma_{ij})n_j d\gamma; \quad \forall \tau \in H_{ad} .$$

Then the *weak* formulation of the plasticity problem is:

Find $\sigma(t) \in P$, $v(t)$ *such that*

$$\begin{cases} A(\dot{\sigma}, \tau - \sigma) + (v, D\tau - D\sigma) \geq \int_{\Gamma_u} \dot{U}_i^d(\tau_{ij} - \sigma_{ij})n_j d\gamma ; \quad \forall \tau \in P \cap H_{ad} \\ (D\sigma, w) + (f, w) = 0 \quad \forall w \in V \\ + \sigma_{ij} n_j = F_i^d \quad \text{on } \Gamma_F \quad \textit{plus initial condition.} \end{cases} \tag{1.8}$$

1.3 Statement of the Results

Proposition 1: *The viscoplastic problem* (1.6) *admits a unique solution* (σ^μ, v^μ), *having the following regularity:*

$$\sigma^\mu \in C^0(0,T; H), \ \dot{\sigma}^\mu \in L^2(0,T;H), \ v^\mu \in L^2(0,T; H^1(\Omega)^N) .$$

Moreover if we assume some additional conditions [3], we will have

Theorem 1: *The perfectly plastic problem* (1.8) *admits a solution* (σ, v), *unique in* σ, *which has the following regularity:*

$$\sigma \in C^0(0,T; H), \ \dot{\sigma} \in L^2(0,T; H), \ v \in L^2_w(0,T; BD(\Omega)) .$$

Also, the perfectly plastic problem is the limit of the viscoplastic problem, i.e. when μ tends to 0:

$$\sigma^\mu \to \sigma \text{ in } L^\infty(0,T; H)$$

$$v^\mu \to v \text{ in } L^2_w(0, T; BD(\Omega)) \quad \text{convergence being in weak-star topology.}$$

1.4 Proof of Theorem 1 (Abbreviated Form)

Proposition 1 is proved by means of the variational formulation (1.6) (see [2,3]). Then taking successively $\tau = \sigma^\mu - x$ and $\tau = \dot{\sigma}^\mu - \dot{x}$ as test functions (x is plastically and statically field the existence of which is ensured by the additional hypothesis), we get the estimates:

σ^μ is a bounded sequence in $L^\infty(0,T; H)$

$\dot{\sigma}^\mu$ is a bounded sequence in $L^2(0,T; H)$

$(J'_\mu(\sigma^\mu), \sigma^\mu - x)$ is a bounded sequence in $L^2(0,T)$.

But the additional hypothesis implies that there exists a ball of radius $\delta > 0$ centered on x and contained in P. Then if $\|\tau\|_\infty \leq \delta$ we have

$$(J'_\mu(\sigma^\mu), \tau) = (J'_\mu(\sigma^\mu), x + \tau - \sigma^\mu) + (J'_\mu(\sigma^\mu), \sigma^\mu - x)$$
$$\leq (J'_\mu(\sigma^\mu), \sigma^\mu - x) .$$

But

$$\delta \| J'_\mu(\sigma^\mu)\|_{(L^1(\Omega))^{N^2}} = \operatorname{Sup}_{\|\tau\|_\infty \leq \delta} (J'_\mu(\sigma^\mu), \tau) .$$

Hence $\varepsilon_{ij}(v^\mu)$ is a bounded sequence in $L^2(0,T; L^1(\Omega))$.

By slight modification of Lemma 3 [3] we conclude:

v^μ is a bounded sequence in $L^2_w(0,T; BD(\Omega))$.

Then, extracting subsequences we prove Theorem 1 and the variational formulation (1.8).

2. REGULARITY OF THE VELOCITY SOLUTIONS

Theorem 2: *Let* Ω *be a regular open set of* R^N, *and* (S) *a subset of* Ω, *locally a* (N - 1) *dimensional manifold, sharing* Ω *in two open and non empty components* Ω_- *and* Ω_+. *Let* $u \in BD(\Omega)$; *then there exists two (unique) elements of* $(L^1(S))^N$, *denoted by* u^- *and* u^+ *satisfying* (2.1), (2.2), *for* $\phi \in (\mathcal{D}(\Omega))^{N^2} \cap H$:

$$\int_{\Omega_-} u_i \phi_{ij,j} dx + \int_{\Omega_-} \phi_{ij} d\varepsilon_{ij}(u) = \int_S u_i^- \phi_{ij} \nu_j d\gamma \tag{2.1}$$

$$\int_{\Omega_-} u_i \phi_{ij,j} dx + \int_{\Omega_-} \phi_{ij} d\varepsilon_{ij}(u) = \int_S u_i^+ \phi_{ij} \nu_j d\gamma ; \tag{2.2}$$

u^- *and* u^+ *are the internal and external traces of* u *on* S *(with respect to* Ω_-*). They are equal in the case of a regular* $((C^0(\bar{\Omega}))^N)$ *function* u.

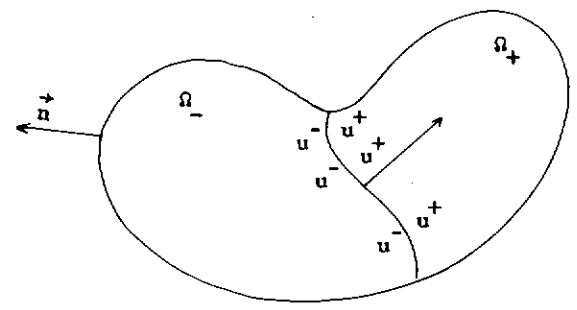

Fig. 1.

Moreover, there exists a unique element u^- *in* $(L^1(\partial\Omega))^N$, *internal trace of* u *on* $\partial\Omega$ *satisfying, for* $\phi \in (\mathcal{D}(\bar{\Omega}))^{N^2} \cap H$:

$$\int_\Omega u_i \phi_{ij,j} dx + \int_\Omega \phi_{ij} d\varepsilon_{ij}(u) = \int_{\partial\Omega} u_i^- \phi_{ij} n_j d\gamma .$$

The application $u \to u^-$ is linear, continuous and surjective from

$BD(\Omega)$ on $(L^1(\partial\Omega))^N$.

2.1 Mathematical Comments

(1) Proof of Theorem 2 can be found in [7] and [8].

(2) Note that u^+ and u^- can be different, due to a possible mass of $\varepsilon_{ij}(u)$ on the surface S.

(3) Note that the notion of trace used in Theorem 2 is weaker than the classical one and is very closed to the notion of right and left limit for functions with bounded variation.

2.2 Mechanical Comments

Elements of $BD(\Omega)$ might present discontinuities. Actually the solution v can be discontinuous:

Example: Unidimensional case: $\Omega =]-1, +1[$, $B(x) = [-1, +1]$

$$A_{ijkh} = +1 , \quad u(1) = u(-1) = 0 , \quad \sigma_0 = u_0 = 0$$
$$f(t,x) = -t(1 - x) \text{ if } x > 0 , \quad f(t, x) = t(x + 1) \text{ if } x < 0 .$$

One can easily check that the solution is elastic till t = 3. Then the solution is

$$\sigma(t, x) = -1 + \begin{cases} -t\,(x + \frac{x^2}{2}) & \text{if } x \leq 0 \\ t(x - \frac{x^2}{2}) & \text{if } x \geq 0 \end{cases}$$

and $$v(t, x) = \begin{cases} -\left(\frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{1}{3} & \text{if } x < 0 \\ -\left(\frac{x^3}{6} - \frac{x^2}{2}\right) - \frac{1}{3} & \text{if } x > 0 \end{cases}$$

which is not continuous at point $x = 0$.

Type of discontinuities: In the general case, discontinuities can be normal and tangential. Normal discontinuities have been encountered by [9] among others. In the case of metals, and of materials with an incompressible plastic rate of strain:

$$\dot{\varepsilon}^{an}_{ii} = 0 \qquad \text{(Von Mises, Tresca criterion...)} ;$$

these discontinuities are only tangential, since we have

$$\mathrm{div}(v) = \mathrm{Trace}\ (A\sigma) \in L^2(\Omega) .$$

Hence $v \cdot \nu$ is continuous across every surface (since well defined in $H^{-1/2}$ of every surface).

For plane problems, the lines of discontinuity are the slip lines (see pictures in [10]).

3. THE BOUNDARY CONDITIONS ON THE VELOCITY FIELD

We have written in the preceding section that every element of $BD(\Omega)$ admits a trace on $\partial\Omega$. The question is: Does this trace v^- satisfy the boundary conditions?

This is not always true: Slip lines, which might appear in the open set, might also appear on the boundary $\partial\Omega$, so that the boundary conditions might not be satisfied (see the preceding example); taking half of the bar and the boundary conditions $v(0) = v(1) = 0$, we get $v(0) = -\frac{1}{3}$).

We propose the following interpretation:

Physical interpretation

The boundary conditions in velocity are imposed by some tool (clamped part, ...) and the plastic material to consider is the union of the body and of the tool. The boundary may be a particular slip line, and the boundary conditions are satisfied by the *external* trace of the velocity and not necessarily by the internal trace.

Hence a plasticity problem must be considered, not only on the open set Ω, but also on a larger set including the boundary $\partial\Omega$.

We are led to the mathematical interpretation:

Mathematical interpretation

We define on $BD(\mathbb{R}^N)$ an equivalence relation:

$$u \mathcal{R} v \iff \begin{cases} u|_\Omega = v|_\Omega \\ \varepsilon_{ij}(u)|_{\bar\Omega} = \varepsilon_{ij}(v)|_{\bar\Omega} \text{ in } M(\bar\Omega) . \end{cases}$$

Then we can check that external trace u^+ and v^+ on $\partial\Omega$ are equal (see (2.2)). We set

$$BD(\bar\Omega) = BD(\mathbb{R}^N)/\mathcal{R}$$

and $BD_{ad}(\bar\Omega) = \{v \mid v \in BD(\bar\Omega);\ v_i^+ = \dot{U}_i^d \text{ on } \Gamma_U\}$.

It can easily be checked that v given by Theorem 1 can be extended to an element of $BD_{ad}(\bar\Omega)$.

$BD_{ad}(\bar\Omega)$ appears to be useful for limit analysis (see in [6] the answer to a paradox of [11]).

Note that the right definition of a kinematically admissible velocity (or displacement) field is a definition on the closed set.

4. CONCLUSION

We shall insist upon the fact that, at the opposite of elasticity problems which can be put on open sets and which admit regular solutions, the plasticity problems need to be put on the closed set occupied by the body, and their solutions can be discontinuous.

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