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LOCKING MATERIALS AND HYSTERESIS PHENOMENA

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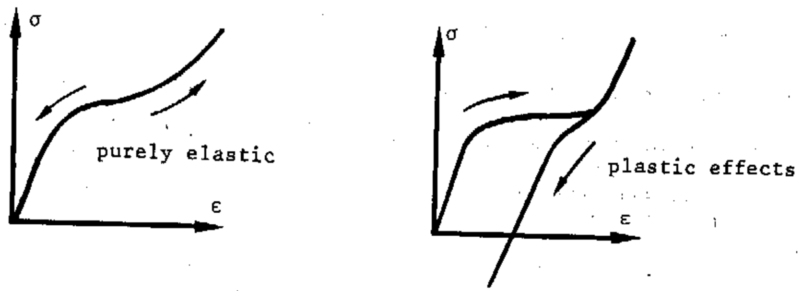
Abstract. A modelling of mechanical hysteresis phenomena, accounting for internal locking of materials is proposed. A mathematical discussion of ideal locking materials is given. A special emphasis is set on the locking limit analysis.

Résumé. On propose un modèle d'hystérésis mécanique, tenant compte des effets de blocage interne de la matière. Le cas des matériaux à blocage est discuté sous un angle mathématique. On porte une attention particulière à l'analyse limite de blocage.

1. SYNOPSIS

Locking materials have been introduced by PRAGER in 1957-1958 in order to account for internal unilateral constraints in the mechanics of continua [1, 2, 3]. For this type of materials the stress-strain curve exhibits an hardening part revealing an internal locking of the matter. This hardening effect can be purely elastic (rubber) or accompanied by plastic effects (cristals). In the last case, hysteresis phenomena similar to those observed in electro-magnetism, take place.



- Figure 1 -

The present work, devoted to a discussion of a few aspects of locking and hysteresis phenomena, is twofold :

- the first part proposes a possible modelling of hysteresis phenomena. Constitutive laws are derived and their structure is discussed. A few open mathematical problems are addressed.

- the second part is devoted to ideal locking materials, as considered by PRAGER. We focus the attention on what is called here the locking limit analysis, the aim of which is to determine the set of admissible imposed displacements before complete locking. The example of torsion of cylindrical bars is discussed : it shows that stress singularities are likely to occur.

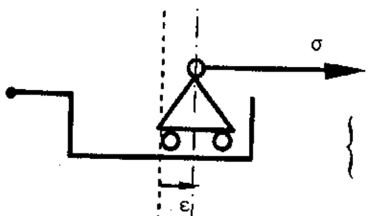
ACKNOWLEDGMENTS

The second part of this work is partly taken from a joint study with F. DEMENGEL [4], whose help is gratefully acknowledged.

2. CONSTITUTIVE LAWS AND MECHANICAL HYSTERESIS

2.1. Rheological models.

The classical rheological models are well known : spring, dash-pot, glider. We introduce a locking model which exhibits the following constitutive law

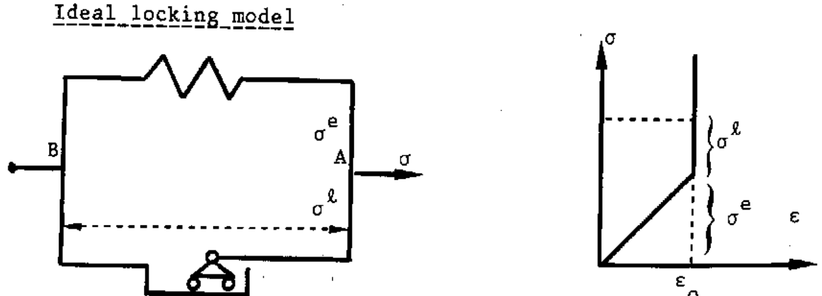


$$|\varepsilon| \leqslant \varepsilon_o \quad \text{and} \quad \begin{cases} |\varepsilon| < \varepsilon_o \Rightarrow \sigma = 0 \\ |\varepsilon| = \varepsilon_o \Rightarrow \sigma = \lambda \dfrac{\varepsilon}{|\varepsilon|} \quad \lambda \geqslant 0 \end{cases}$$

- Figure 2 -

This element called a *lock*, is used in more complex models.

Ideal locking model



- Figure 3 -

The constitutive law of ideal locking materials is :

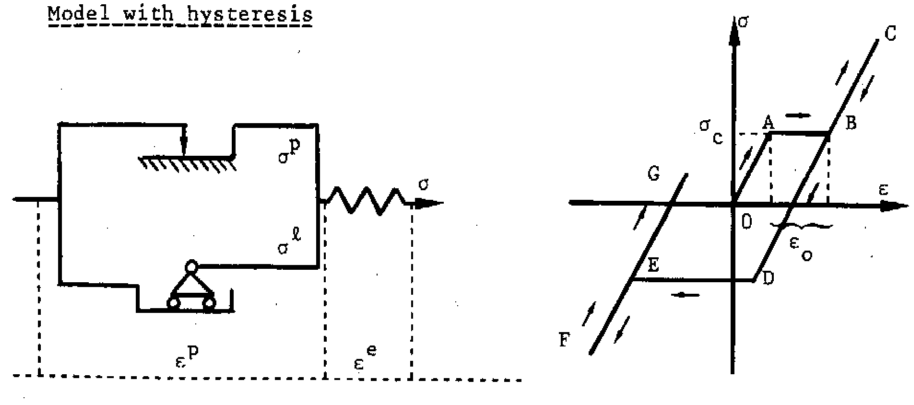
$$\begin{cases} \sigma = \sigma^e + \sigma^\ell \quad , \quad \sigma^e = E\,\varepsilon \\ |\varepsilon| \leqslant \varepsilon_o \ , \ \sigma^\ell = 0 \ \text{if} \ |\varepsilon| < \varepsilon_o \ , \ \sigma^\ell = \lambda \dfrac{\varepsilon}{|\varepsilon|} \ \text{if} \ |\varepsilon| = \varepsilon_o \ , \ \lambda \geqslant 0 \end{cases} \tag{2.1}$$

Instead of imposing a given stress in A , one can impose a given displacement u^d . This given displacement must obey

$$|u^d| \leqslant \varepsilon_o$$

Therefore, a locking material cannot undergo *any* imposed displacement. The determination of the admissible imposed displacements is the object of the *locking limit analysis* (cf. § 3.2) .

Model with hysteresis



a. Model b. Loading-Unloading Test

- Figure 4 -

The constitutive law of the model is the following

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad \sigma = \sigma^p + \sigma^\ell$$

$$\varepsilon^e = \frac{\sigma}{E} ,$$

$$|\sigma^p| \leq \sigma_c^{(\dagger)} \text{ and } \begin{cases} \dot\varepsilon^p = 0 & \text{if } |\sigma^p| < \sigma_c \\ \dot\varepsilon^p = \lambda \dfrac{\sigma^p}{|\sigma^p|} & \text{if } |\sigma^p| = \sigma_c , \ \lambda \geq 0 \end{cases} \tag{2.2}$$

$$|\varepsilon^p| \leq \varepsilon_o \text{ and } \begin{cases} \sigma^\ell = 0 & \text{if } |\varepsilon^p| < \varepsilon_o \\ \sigma^\ell = \lambda' \dfrac{\varepsilon^p}{|\varepsilon^p|} & \text{if } |\varepsilon^p| = \varepsilon_o , \ \lambda' \geq 0 \end{cases}$$

We investigate the behavior of the model in a loading-unloading experiment :

- OA : The spring is the only strained element
- AB : Gliding without elevation of the applied stress
- BC : locking of the lock : the spring is the only strained element
- CBD : The glider is locked : the unloading is purely elastic until the stress reaches $-\sigma_c$
- DE : The glider slips without stress modification

(†) σ_c is the yield limit of the glider.

- EF : locking of the lock
- FEG : The glider is fixed

The model exhibits an *hysteresis behavior.*

2.2. 3-dimensional case.

Ideal locking material

The natural generalization of (2.1) goes as follows :

There exists a convex set B in the strain space, which the strain tensor is constrained to stay in :

$$\varepsilon(u) \in B .$$

Moreover

$$\begin{cases} \sigma_{ij} = \sigma^e_{ij} + \sigma^\ell_{ij} \\ \sigma^e_{ij} = a_{ijkh}\, \varepsilon_{kh}(u) \\ \sigma^\ell \in \partial I_B(\varepsilon(u)) \end{cases} \tag{2.3}$$

where I_B is the indicator function of the set B in the space E of 3 × 3 symmetric tensors of order 2 .

Remark . A typical example of set B is the following

$$B = \{e \in E \mid -k_1 \leq e_{ii} \leq k_o\}$$

Only volumic changes are constrained. The class of locking materials described by this choice of B is that of materials with limited compressibility. The case of incompressible materials is recovered with a special choice $k_o = k_1 = 0$.

Locking materials with hysteresis

The natural generalization of (2.2) goes as follows

$$\varepsilon_{ij}(u) = \varepsilon^e_{ij} + \varepsilon^p_{ij} , \qquad \sigma_{ij} = \sigma^p_{ij} + \sigma^\ell_{ij}$$

$$\varepsilon^e_{ij} = \Lambda_{ijkh}\, \sigma_{kh}$$

There exists a convex set B in the strain space ($\simeq E$) which the plastic strain tensor is constrained to stay in

$$\varepsilon^p \in B \qquad \sigma^\ell \in \partial I_B(\varepsilon^p) \qquad B \text{ closed convex set in } E$$

There exists a convex set P in the stress space ($\simeq E$) which the first part of the stress tensor is constrained to stay in

$$\sigma^p \in P \quad , \quad \dot\varepsilon^p \in \partial I_P(\sigma^p) \qquad P \text{ closed convex set in } E$$

Therefore the constitutive law, written in a condensed form, amounts to :

$$\left\{\begin{array}{lll} \varepsilon = \varepsilon^e + \varepsilon^p & , \ \sigma = \sigma^p + \sigma^\ell & , \ \varepsilon^e = \Lambda\sigma \\ \varepsilon^p \in B & , \ \sigma^\ell \in \partial I_B(\varepsilon^p) & \\ \sigma^p \in P & , \ \dot\varepsilon^p \in \partial I_P(\sigma^p) \ . & \end{array}\right. \qquad (2.4)$$

We claim that the two constitutive laws have the same structure : they both are *generalized standard materials.*

2.3. Generalized standard materials.

The theory of generalized standard materials, due to HALPHEN, NGUYEN QUOC SON takes its roots into ZIEGLER's and MOREAU's works. It proposes a general framework for the establishment of constitutive laws, accounting for the two laws of Thermodynamics (detailed exposures can be found in HALPHEN, NGUYEN QUOC SON [5], NGUYEN QUOC SON [6], GERMAIN [7], GERMAIN, NGUYEN QUOC SON and SUQUET [8], SUQUET [9]) .

We admit the existence of a density of free energy depending on the state variables $(\varepsilon,\alpha)^{(\dagger)}$:

$$\rho w = \rho w(\varepsilon,\alpha) \qquad (\rho \text{ density of the body})$$

ρw is supposed to be convex with respect to (ε,α) .

The *state laws* define the thermodynamical forces

$$\sigma^R = \rho\, \frac{\partial w}{\partial \varepsilon}(\varepsilon,\alpha) \quad , \qquad A = -\,\rho\, \frac{\partial w}{\partial \alpha}(\varepsilon,\alpha)$$

σ^R is the reversible part of the stress tensor.

In case of a nondifferentiable free energy w the preceeding relations are to be understood in the sense of subdifferentials [10], [11]

(†) for the sake of simplicity we omit thermal effects. The temperature T will not be listed among the state variables.

$$(\sigma^R, -A) \in \rho\partial w(\varepsilon,\alpha) \tag{2.5}$$

We admit the existence of a potential of dissipation $\mathcal{D}$ convex function of its arguments $(\dot\varepsilon,\dot\alpha)$, which yields the *complementary laws*

$$\sigma^{IR} = \sigma - \sigma^R = \frac{\partial \mathcal{D}}{\partial \dot\varepsilon}(\dot\varepsilon,\dot\alpha) \quad , \quad A = \frac{\partial \mathcal{D}}{\partial \dot\alpha}(\dot\varepsilon,\dot\alpha)$$

or in a generalized sense :

$$(\sigma^{IR}, A) \in \partial\mathcal{D}(\dot\varepsilon,\dot\alpha) \tag{2.6}$$

σ^{IR} is the irreversible part of the stress tensor.

In the framework of generalized standard materials *a constitutive law is specified by the data of the two thermodynamical potentials* ρw *and* $\mathcal{D}$.

It can be proved [6, 7, 9] that the *mechanical dissipation* amounts to

$$d = \sigma^{IR}\dot\varepsilon + A\dot\alpha$$

Application to the specific situation of locking materials.

Ideal locking materials

The only state variable is the strain ε. The two thermodynamical potentials amount to

$$\begin{cases} \rho w(\varepsilon) = \frac{1}{2} a_{ijkh}\varepsilon_{kh}\varepsilon_{ij} + I_B(\varepsilon) \\ \mathcal{D}(\dot\varepsilon) = 0 \end{cases}$$

The state laws (2.5) and the complementary laws (2.6) yield

$$\begin{cases} \sigma^R \in a\varepsilon + \partial I_B(\varepsilon) \\ \sigma^{IR} = \sigma - \sigma^R = 0 \end{cases}$$

which is exactly (2.3).

Remark. The ideal locking material is *not dissipative* ($\mathcal{D} = 0$) : it is an *hyperelastic* material.

Locking materials with hysteresis

The state variables are ε and $\alpha = \varepsilon^P$. The following choice of potentials is made

$$\begin{cases} \rho w(\varepsilon,\varepsilon^P) = \frac{1}{2} a(\varepsilon - \varepsilon^P)(\varepsilon - \varepsilon^P) + I_B(\varepsilon^P) \\ \mathcal{D}(\dot\varepsilon,\dot\varepsilon^P) = I_P^*(\dot\varepsilon^P) \ (\dagger) \end{cases}$$

The state laws (2.5) yield

$$\sigma^R = a(\varepsilon - \varepsilon^P) \quad , \quad A \in a(\varepsilon - \varepsilon^P) - \partial I_B(\varepsilon^P) \tag{2.7}$$

The complementary laws (2.6) yield

$$\sigma^{IR} = 0 \quad , \quad A \in \partial(I_P^*)(\dot\varepsilon^P) \quad \text{i.e.} \quad \dot\varepsilon^P \in \partial I_P(A) \tag{2.8}$$

Therefore the total stress reduces to the reversible stress

(†) I_P^* denotes the Legendre Fenchel transform of I_P [10,11].

$$\sigma = \sigma^R = a(\varepsilon - \varepsilon^p)$$

Setting

$$\varepsilon^e = \varepsilon - \varepsilon^p \ , \quad \Lambda = a^{-1} \ , \ \sigma^p = A \ , \ \sigma^\ell = a(\varepsilon - \varepsilon^p) - \sigma^p \in \partial I_B(\varepsilon^p)$$

we see that the law defined by (2.7)(2.8) takes the form (2.4) . The locking material under consideration here is *dissipative*, and the mechanical dissipation amounts to

$$d_1 = \sigma^{IR}\,\dot{\varepsilon} + A\dot{\alpha} = \sigma^p\,\dot{\varepsilon}^p \ .$$

Remark. ZIEGLER and PRAGER [2] also considered a non newtonian fluid for which the locking constraints acts on the *strain rate* $\dot{\varepsilon}$. This type of fluid is also a generalized standard material. The choice of state variables and potentials goes as follows

state variables : ε

potentials : $\begin{cases} \rho w(\varepsilon) = \Phi(Tr\varepsilon) \\ \mathcal{D}(\dot{\varepsilon}) = I_B(\dot{\varepsilon}) \end{cases}$

the constitutive law amounts to

$$\sigma^R = - p\ Id \qquad \text{where} \qquad p = - \frac{\partial \Phi}{\partial(Tr\varepsilon)}$$

$$\sigma^{IR} \in \partial I_B(\dot{\varepsilon})$$

i.e. $\sigma \in -p\ Id + \partial I_B(\dot{\varepsilon})$.

Remark. The interest of recognizing a generalized standard form in a constitutive law is that general theorem on variational principles,

behavior at infinity... have been derived in this general setting.

2.4. Evolution problem for a locking material with hysteresis.

We now turn to the evolutive boundary problem posed by a locking material occupying a bounded domain Ω , submitted to body and boundary forces, to imposed displacements, and undergoing a quasi-static evolution.

In addition to the constitutive law (2.4) the stress and strain fields must obey further requirements

$$\begin{cases} \varepsilon_{ij} = \varepsilon_{ij}(u) & \text{in } \Omega \quad \text{compatibility relations} \\ \dfrac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = 0 & \text{in } \Omega \quad \text{equilibrium equations} \\ u_i = U_i^d & \text{on } \partial\Omega_U \quad \text{imposed displacements on a part } \partial\Omega_U \text{ of } \partial\Omega \\ \sigma_{ij} n_j = F_i^d & \text{on } \partial\Omega_F \quad \text{imposed forces on a part } \partial\Omega_F \text{ of } \partial\Omega^{(\dagger)} \end{cases}$$

The loading $f(x,t)$, $F^d(x,t)$, $U^d(x,t)$ is given on $[0,T]$.

Splitting of the problem.

We first solve a purely elastic problem :

$$\begin{cases} \varepsilon_{ij}(u^{e\ell}) = \Lambda_{ijkh}\,\sigma^{e\ell}_{kh} \ , \ \dfrac{\partial \sigma^{e\ell}_{ij}}{\partial x_j} + \rho f_i = 0 \\ \sigma^{e\ell}_{ij}\,n_j = F_i^d \ \text{ on } \partial\Omega_F \ , \quad u_i^{e\ell} = U_i^d \ \text{ on } \partial\Omega_U \end{cases}$$

Provided that :

. Ω is a bounded domain with a Lipschitz boundary

(†) $\partial\Omega_U$, $\partial\Omega_F$ are open and disjoint subsets of $\partial\Omega$ $\overline{\partial\Omega_U} \cup \overline{\partial\Omega_F} = \partial\Omega$

. $f \in W^{1,2}(0,T\ ;\ L^2(\Omega)^3)$, $F^d \in W^{1,2}(0,T\ ;\ L^2(\partial\Omega)^3)$,
$U^d \in W^{1,2}(0,T\ ;\ H^{1/2}(\partial\Omega)^3)$

. the elasticity matrix is symmetric, bounded and coercive,

the above elastic problem admits a solution $(\sigma^{e\ell}, u^{e\ell})$

$\sigma^{e\ell} \in W^{1,2}(0,T\ ;\ L^2(\Omega,E))$ (†)

$u^{e\ell} \in W^{1,2}(0,T)\ ;\ \mathbb{H}^1(\Omega))$ (††)

Setting

$$\bar{\sigma} = \sigma - \sigma^{e\ell} \qquad \text{and} \qquad \bar{u} = u - u^{e\ell}$$

we see that $\bar{\sigma}$, $\bar{u}$ satisfies the following set of equations .

$$\begin{cases} \varepsilon(\bar{u}) = A\bar{\sigma} + \varepsilon^P \ , \quad \operatorname{div} \bar{\sigma} = 0 \quad \text{in} \quad \Omega \\ \bar{\sigma}.n = 0 \quad \text{on} \quad \partial\Omega_F \ , \quad \bar{u} = 0 \quad \text{on} \quad \partial\Omega_U \end{cases}$$

For a given ε^P in $L^2(\Omega,E)$ we can associate the unique solution $\bar{\sigma}$ in $L^2(\Omega,E)$ of the preceding elastic problem :

$$\bar{\sigma} = -\ R\ \varepsilon^P$$

This equality defines a *linear, continuous self adjoint and maximal monotone operator* R *from* $L^2(\Omega,E)$ *into itself.*

From the definition of $\bar{\sigma}$ we derive

(†) $L^2(\Omega,E)$ = symmetric 3×3 tensors of order 2 with components in $L^2(\Omega)$

(††) $\mathbb{H}^1(\Omega) = H^1(\Omega)^3$

$$\sigma(t) = -\ R\ \varepsilon^P(t) + \sigma^{e\ell}(t) \qquad \text{in} \quad L^2(\Omega,E)$$

Therefore *the determination of the stress field reduces to the determination of the field of plastic strains.* (†)

Evolution equation for the plastic strains

We note that

$$\sigma(t) = \sigma^P(t) + \sigma^\ell(t) \qquad \text{in} \quad L^2(\Omega,E)$$

where $\quad \sigma^\ell(t) \in \partial I_{\mathbb{B}}(\varepsilon^P(t))$ "

and $\quad \sigma^P(t) \in \partial(I^\star_{\mathbb{P}})(\dot{\varepsilon}^P(t))$ "

$\mathbb{B}$ and $\mathbb{P}$ are respectively defined as

$$\mathbb{B} = \{e \in L^2(\Omega,E)\ ,\ e(x) \in B \quad \text{a.e.}x \in \Omega\}$$

$$\mathbb{P} = \{\tau \in L^2(\Omega,E)\ ,\ \tau(x) \in P \quad \text{a.e.}x \in \Omega\}$$

Therefore the field of plastic strains satisfies the following *non-linear evolution equation* in $L^2(\Omega,E)$

$$\begin{cases} \partial(I^\star_{\mathbb{P}})(\dot{\varepsilon}^P(t)) + \partial I_{\mathbb{B}}(\varepsilon^P(t) + R\ \varepsilon^P(t) \ni \sigma^{e\ell}(t) \\ \varepsilon^P(0) = \varepsilon^P_o \end{cases} \tag{2.9}$$

(†) $\sigma^{e\ell}(t)$ is a given quantity.

The resolution of this evolution equation seems to be an open problem. It bears some ressemblance with an equation discussed by VISINTIN [12] in a problem issued from phase transition, but where compactness methods applied (this is not the case here).

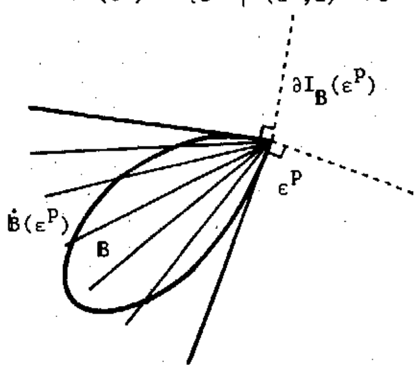
This problem being too difficult, we turn to an easier one.

Rate principles for the plastic strains

Let us assume that the present state of plastic strain $\varepsilon^P(t)$ is known. We want to determine the *rate* $\dot{\varepsilon}^P(t)$ of plastic strain.

Since ε^P must belong to $\mathbb{B}$, the rate $\dot{\varepsilon}^P$ is constrained to stay in the projecting cone of $\mathbb{B}$ at $\varepsilon^P(t)$

$$\dot{\mathbb{B}}(\varepsilon^P) = \{\varepsilon^\star \mid (\varepsilon^\star, z) \leq 0 \quad \forall\, z \in \partial I_{\mathbb{B}}(\varepsilon^P)\}$$



A field $\varepsilon^\star$ belonging to $\dot{\mathbb{B}}(\varepsilon^P)$ is said to be a *locking rate of plastic strain.*

\- Figure 5 -

Variational principle for the strain rate $\dot{\varepsilon}^P$

$\dot{\varepsilon}^P$ minimizes among all locking admissible rates of plastic strains the functional

$$\operatorname*{Inf}_{\varepsilon^\star \in \dot{\mathbb{B}}(\varepsilon^P(t))} \quad I^\star_{\mathbb{P}}(\varepsilon^\star) + (R\varepsilon^P(t), \varepsilon^\star)_{L^2} - (\sigma^{e\ell}(t), \varepsilon^\star)_{L^2} \tag{2.10}$$

Proof. We notice that

$$(\varepsilon^\star, z)_{L^2} \leq 0 \qquad \forall\, z \in \partial I_{\mathbb{B}}(\varepsilon^P(t))$$

we shall use this inequality with $z = \sigma^\ell$

On the other hand

$$(\dot{\varepsilon}^P, \sigma^\ell)_{L^2} = \frac{d}{dt}(I_{\mathbb{B}}(\varepsilon^P)) = \frac{d}{dt}\, 0 = 0$$

Therefore

$$(\sigma, \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} = (\sigma^P, \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} + (\sigma^\ell, \varepsilon^\star - \dot{\varepsilon}^P)_{L^2}$$

$$\leq (\sigma^P, \varepsilon^\star - \dot{\varepsilon}^P)_{L^2}$$

But $(\sigma, \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} = -(R\varepsilon^P(t), \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} + (\sigma^{e\ell}(t), \varepsilon^\star - \dot{\varepsilon}^P)_{L^2}$

Thus

$$I^\star_{\mathbb{P}}(\varepsilon^\star) - I^\star_{\mathbb{P}}(\dot{\varepsilon}^P) + (R\varepsilon^P(t), \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} - (\sigma^{e\ell}(t), \varepsilon^\star - \dot{\varepsilon}^P)_{L^2}$$

$$\geq (\sigma^P, \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} + (R\varepsilon^P(t), \varepsilon^\star - \dot{\varepsilon}^P)_{L^2} - (\sigma^{e\ell}(t), \varepsilon^\star - \dot{\varepsilon}^P)_{L^2}$$

$$\geq 0$$

Remark. Due to the high nonlinearity of the thermodynamical potentials, the variational principles established in NGUYEN QUOC SON [13] or SUQUET [9] do not apply.

2.5. Open problems.

. Under which set of assumptions on the loading is it possible to prove the existence of a solution of the variational principle (2.10) ?

. Same question for the evolution equation (2.9) .

. Consider a proportional loading

$$\lambda f \ , \ \lambda F^d \ , \ \lambda U^d$$

We can easily conceive that for small λ equation (2.9) admits a solution. Does there exist a limit value for λ (similar to the plastic limit load, or the locking limit load) ?

. Is it possible to discuss under general assumptions on the functionals φ and ψ , the following evolution equation :

$$\begin{cases} \partial\varphi(\dot{\alpha}) + \partial\psi(\alpha) + R\alpha \ni f(t) & \text{in a Hilbert space } H \\ \alpha(0) = \alpha_o \end{cases}$$

where φ and ψ are convex, l.s.c., proper functionals on H , R is a linear, positive continuous operator from H into H .

3. IDEAL LOCKING MATERIALS

We consider the hyperelastic materials introduced originally by PRAGER, namely ideal locking materials. Let us recall the constitutive law (2.3) :

$$\begin{cases} \sigma_{ij} = \sigma^e_{ij} + \sigma^\ell_{ij} \\ \sigma^e_{ij} = a_{ijkh}\,\varepsilon_{kh}(u) \\ \varepsilon(u) \in B \\ (\sigma^\ell_{ij}, \varepsilon^\star_{ij} - \varepsilon_{ij}(u)) \leqslant 0 \qquad \forall\ \varepsilon^\star \in B \end{cases} \tag{3.1}$$

The free energy of the material reduces to its elastic energy

$$\rho w(\varepsilon) = \begin{cases} \frac{1}{2} a_{ijkh}\,\varepsilon_{ij}\,\varepsilon_{kh} & \text{if } \varepsilon \in B \\ +\infty & \text{otherwise} \end{cases}$$

A few assumptions on material datas (†) ensure that ρw is a convex

(†) B is bounded and contains 0 as an interior point function, continuous on the interior of its domain ; moreover its conjugate function $\rho w^{\star}$ satisfies

$$\exists c_o, c_1 > 0 \ , \ \forall \ \xi \in E \ , \ c_o(|\varepsilon|_E - 1) \leq \rho w^{\star}(\varepsilon) \leq c_1(|\varepsilon|_E + 1) \tag{3.2}$$

The constitutive law (3.1) now reads

$$\begin{cases} \varepsilon(u(x)) \in B \\ \sigma(x) \in \partial \rho w^{\star}(\varepsilon(u(x))) \ , \quad \text{a.e.}x \ \text{ in } \ \Omega \end{cases}$$

W and $W^{\star}$ denote the strain energy and the complementary energy of the body respectively defined on $L^2(\Omega,E)$ and $L^1(\Omega,E)$ as

$$W^{\star}(\sigma) = \int_{\Omega} \rho w^{\star}(\sigma(x))dx$$

$$W(\varepsilon) = \int_{\Omega} \rho w(\varepsilon(x))dx$$

In addition to the constitutive law, the stress and strain must satisfy the following requirements :

$$\begin{cases} \dfrac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = 0 \quad \text{in } \Omega \quad \text{equilibrium equations} \\ \sigma_{ij} \, n_j = F_i^d \quad \text{on } \partial\Omega_F \\ u_i = U_i^d \quad \text{on } \partial\Omega_U \end{cases}$$

A few assumptions on the loadings (†) enable us to define the spaces

(†) $\rho f \in L^2(\Omega)^2$, $F^d \in L^2(\partial\Omega)^3$, $U^d \in H^{1/2}(\partial\Omega)^3$

of kinematically admissible fields of statically admissible fields, and the work of given external forces as :

$$\mathcal{U}_{ad} = \{u \in \mathbb{H}^1(\Omega) \ , \ u = U^d \ \text{ on } \ \partial\Omega_U\}$$

$$\mathcal{B} = \{u \in \mathbb{H}^1(\Omega) \ , \ \varepsilon(u(x)) \in B \quad \text{a.e.}x \in \Omega\}$$

$$S_{ad} = \{\sigma \in L^2(\Omega,E) \ , \ \text{div}\,\sigma + \rho f = 0 \ \text{ in } \ \Omega \ , \ \sigma.n = F^d \ \text{ on } \ \partial\Omega_F\}$$

$$L(u) = \int_{\Omega} \rho f u \, dx + \int_{\partial\Omega_F} F^d u \, ds \qquad \forall \ u \in \mathbb{H}^1(\Omega) \ .$$

3.1. Variational properties of the stress and displacement fields.

Provided that they have a minimum regularity, the solutions (σ,u) of the above problem satisfy the following variational principles (PRAGER [2], DUVAUT-LIONS [14]) in terms of the *strain energy* :

$$\inf_{u \in \mathcal{U}_{ad}} [W(\varepsilon(u)) - L(u)] \ , \tag{3.3}$$

in terms of the *complementary energy* :

$$\sup_{\sigma \in S_{ad}} [- W^{\star}(\sigma) + \int_{\partial\Omega_U} \sigma.n \, U^d \, ds] \tag{3.4}$$

Problem (3.3) *admits a solution*, provided that the following condition is satisfied :

$$\mathcal{B} \cap \mathcal{U}_{ad} \neq \emptyset \tag{3.5}$$

Indeed, W turns out to be a strictly convex lower semi continuous and coercive functional :

$$W(\varepsilon(u)) \geq \frac{\alpha}{2} \; |\varepsilon(u)|_2^2$$

Korn's inequality proves the coercivity on the non empty closed convex subset $\mathcal{B} \cap \mathcal{U}_{ad}$ of $\mathbb{H}^1(\Omega)$ of the functional involved in (3.3). Then, existence and uniqueness of a solution for the variational problem (3.3) is easily derived.

The variational problem (3.4) for the stress field σ is a more difficult one. Indeed, as a consequence of (3.2), the functional W^* is only coercive on $L^1(\Omega,E)$ which is a non reflexive space. Therefore, proving the existence of a solution of (3.4) by means of classical arguments of coercivity, requires the introduction of a new functional space, accounting for *stress concentrations* :

$$\Sigma(\Omega) = \{\sigma \in M^1(\Omega,E)\,,\, \operatorname{div}\sigma \in L^2(\Omega,\mathbb{R}^N)\} \qquad (M^1 = \text{bounded measures})$$

A detailed study of $\Sigma(\Omega)$ can be found in [15], and the proof of the existence of a solution σ in $\Sigma(\Omega)$ is completed in [16]. In order that the solutions u and σ of (3.3) and (3.4) satisfy the extremality relations, i.e. the constitutive law (3.1) it is necessary [10] that

$$\operatorname{Inf}(3.3) = \operatorname{Sup}(3.4) \tag{3.6}$$

The proof of (3.6) can be found in DEMENGEL-SUQUET [4].

3.2. Locking limit analysis.

In the previous works concerned with locking materials the assumption

$$\mathcal{B} \cap \mathcal{U}_{ad} \neq \phi \tag{3.7}$$

was not discussed. By analogy to the plastic limit analysis we call such a discussion the *locking limit analysis* : it determines for which set of imposed displacements U^d, the condition (3.7) is fullfilled.

For sake of simplicity, U^d is assumed to be proportional to a load parameter

$$U^d = \lambda u_o$$

The space of kinematically admissible fields now depends on the load parameter

$$\mathcal{U}_{ad}(\lambda) = \{u \in \mathbb{H}^1(\Omega)\,,\; u = \lambda u_o \text{ on } \partial\Omega_U\}$$

The locking limit analysis amounts to determine the admissible values of λ. Let us define

$$\overline{\lambda}_\ell = \operatorname{Sup}\{\lambda \in \mathbb{R} \mid \mathcal{B} \cap \mathcal{U}_{ad}(\lambda) \neq \phi\} \qquad \textit{Problem } Q \tag{3.8}$$

Since a sup is computed in (3.8) $\overline{\lambda}_\ell$ is approximated by *lower values*. An approximation by *upper values* can be proposed which consists of the dual problem of (3.8) :

$$\inf_{\sigma \in S_o} \int_\Omega \pi_B(\sigma)\,dx \qquad \textit{Problem } Q^\star \tag{3.9}$$

$$\int_{\partial\Omega_U} \sigma.n\, u_o\, ds = 1$$

where

$$S_o = \{\sigma \in L^2(\Omega,E)\ ,\ \text{div}\ \sigma = 0 \quad \text{a.e. in } \Omega\ ,\ \sigma.n = 0 \text{ on } \partial\Omega_F\}$$

$$\pi_B(\sigma) = \sup_{e \in B} (\sigma,e)\ .$$

We shall prove the following theorem :

THEOREM 2. Under the assumptions H1,H2 *listed hereafter :* $Q^\star$(3.9) *is the dual problem of* Q (3.8) *and the primal dual relations hold :*

$$\text{Sup } Q = \text{Inf } Q^\star$$

If moreover H3 *is satisfied :*

$$\overline{\lambda}_\ell = \text{Inf } Q^\star = \text{Sup } Q < +\infty \tag{3.10}$$

$\overline{\lambda}_\ell$ *is the locking limit load .*

Remark. It is worth noting the analogy with the plastic limit analysis. Problem (3.8) is a kinematical approach of $\overline{\lambda}_\ell$ while (3.9) is a statical approach of $\overline{\lambda}_\ell$.

H1 Ω is a bounded domain with a Lipschitz boundary. B is a bounded, closed convex set of E containing 0 as an interior point.

H2 . $\rho f \in L^2(\Omega)^3$, $F^d \in L^2(\partial\Omega)^3$

. $u_o \in W^{1,\infty}(\partial\Omega)^N$ and $u_o = \chi_o u_o$

where χ_o denotes the characteristic function of $\partial\Omega_U$.

H3 There exists $\sigma^o \in S_o$ such that

$$\int_{\partial\Omega_U} \sigma^o.n\, u_o\, ds \neq 0\ .$$

Proof. The proof requires several steps. We shall complete the first one ; the other ones are treated in full details in [4] .

We set

$$V = H^1(\Omega)\ ,\quad Y = L^2(\Omega,E)$$

We define a linear operator $\Lambda \in \mathcal{L}(V,Y)$ by

$$\Lambda v = \varepsilon(v)\ .$$

We define on V and Y two functionals F and G by :

$$F(v) = \begin{cases} -\lambda & \text{if } u = \lambda u_o \text{ on } \partial\Omega_U \quad (u \in U_{ad}(\lambda)) \\ +\infty & \text{otherwise} \end{cases}$$

$G(p) = I_{\mathbb{B}}(p)$ where $I_{\mathbb{B}}$ denotes the indicator function of $\mathbb{B}$.

Problem (3.8) now reads :

$$\inf_{v \in V} \{F(v) + G(\Lambda v)\}$$

and its dual problem is [5] :

$$\cdot \sup_{p^\star \in Y^\star} \{- G^\star(-p^\star) - F^\star(\Lambda^\star p^\star)\}$$

A theorem of KRASNOSELSKII ensures that :

$$G^\star(p^\star) = \int_\Omega \pi_B(p^\star)dx .$$

The computation of $F^\star(\Lambda^\star p^\star)$ is performed in the following proposition.

Proposition.

$$F^\star(\Lambda^\star p^\star) = \begin{cases} 0 & \textit{if} \quad p^\star \in S_o \quad \textit{and} \quad \int_{\partial\Omega_U} p^\star.n\, u_o\, ds = 1 \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof.

$$F^\star(\Lambda^\star p^\star) = \sup_{u \in \mathcal{U}_{ad}(\lambda)} [(\Lambda^\star p^\star, u) + \lambda] \qquad (3.11)$$

From assumption H2 we deduce that there exists $U_o \in W^{1,\infty}(\Omega)^N$ such that

$$U_o = u_o \quad \text{on} \quad \partial\Omega \quad , \quad \text{i.e.} \quad \lambda U_o \in \mathcal{U}_{ad}(\lambda)$$

Then :

$$F^\star(\Lambda^\star p^\star) \geq \sup_{\lambda \in \mathbb{R}} \lambda [(\Lambda^\star p^\star, U_o) + 1]$$

This supremum equals $+\infty$ except when

$$(\Lambda^\star p^\star, U_o) + 1 = 0 \qquad (3.12)$$

We perform the computation of $F^\star(\Lambda^\star p^\star)$ in this specific situation. Using the fact that $1 = -(\Lambda^\star p^\star, U_o)$ we get from (3.12) :

$$F^\star(\Lambda^\star p^\star) = \sup_{u \in \mathcal{U}_{ad}(\lambda)} [(\Lambda^\star p^\star, u - \lambda U_o)]$$

$$= \sup_{\substack{v \in V \\ v = 0 \text{ on } \partial\Omega_U}} (\Lambda^\star p^\star, v)$$

The computation of this last supremum is classical (TEMAM [17]) : $F^\star(\Lambda^\star p^\star) = +\infty$ except for the $p^\star$ satisfying :

$$\operatorname{div} p^\star = 0 \text{ in } \Omega , \quad p^\star.n = 0 \text{ on } \partial\Omega_F \qquad (3.13)$$

With the help of Green's formula the condition (3.12) is equivalent to :

$$\int_{\partial\Omega_U} p^\star.n\, u_o\, ds + 1 = 0$$

Finally we have shown that a necessary condition to be satisfied by $p^\star$ in order to give a finite value to $F^\star(\Lambda^\star p^\star)$ is :

$$p^\star \in S_o \quad , \quad \int_{\partial\Omega_U} p^\star.n\, u_o\, ds + 1 = 0 \qquad (3.14)$$

Under these conditions it can be proved from (3.11)(†) that

$F^{\star}(\Lambda^{\star} p^{\star}) = 0$ for $p^{\star}$ satisfying (3.14)

which completes the proof of the proposition.

It enables us to perform the computation of $Q^{\star}$ which amounts to :

$$- \sup_{p^{\star} \in S_o} \left[- \int_{\Omega} \pi_B(-p^{\star})\,dx \right]$$

$$\int_{\partial\Omega_U} p^{\star}.n\, u_o\, ds + 1 = 0$$

or equivalently

$$\inf_{\sigma \in S_o} \left[\int_{\Omega} \pi_B(\sigma)\,dx \right]$$

$$\int_{\partial\Omega_U} \sigma.n\, u_o\, ds = 1$$

which is exactly (3.9) . The primal-dual relations yield :

$$\sup Q \leq \inf Q^{\star}$$

The proof of the reverse inequality is a technical one. It is due to F. DEMENGEL and uses a penalty method. It can be found in DEMENGEL-SUQUET [4] .

Let us emphasize once more that the stress problem (3.9) is *not*

(†) The complete justification of this point follows the appendix of section 2 in TEMAM [17] .

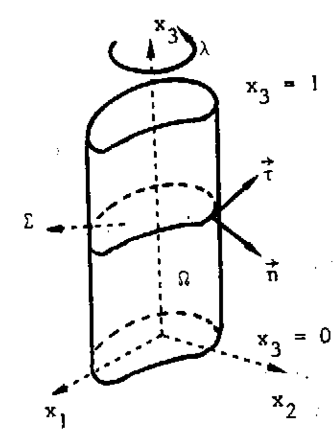
coercive in the classical space $L^2(\Omega,E)$ but only in $L^1(\Omega,E)$. Therefore we expect to find a stress field σ in $M^1(\Omega,E)$. The following example illustrates this point.

3.3. An example of locking limit analysis : torsion of cylindrical bars.

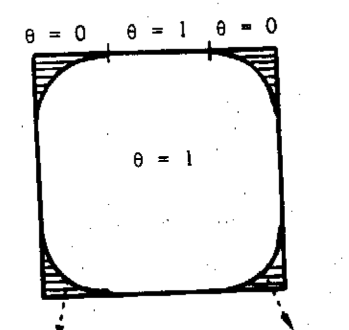
Let us consider a cylindrical bar, with a simply connected cross-section, made from a locking material, and submitted to a torsion experiment, with angle λ . The stress tensor and the displacement field exhibit the following classical form :

$$\sigma = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{pmatrix} , \quad u = \begin{cases} u_1 = -\lambda x_2 x_3 \\ u_2 = \lambda x_1 x_3 \\ u_3 = u_3(x_1, x_2, \lambda) \end{cases}$$

where $\sigma_{ij} = \sigma_{ij}(x_1, x_2)$.



cylindrical bar Ω



square cross-section Σ

- Figure 6 -

Equilibrium equations reduce to

$$\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} = 0$$

The boundary conditions amount to

$$\sigma_{ij}\, n_j = 0 \quad \text{on lateral sides of the cylinder} \tag{3.15}$$

$$\begin{cases} u_1 = -\lambda\, x_2 & \\ u_2 = \lambda\, x_1 & \text{on the upper section of the cylinder,} \quad x_3 = 1\,. \\ \sigma_{33} = 0 & \\ u_1 = u_2 = 0 & \text{on the lower section of the cylinder,} \quad x_3 = 0\,. \\ \sigma_{33} = 0 & \end{cases}$$

A classical analysis of the equilibrium equations shows the existence of a *stress function* θ such that

$$\sigma_{13} = \frac{\partial\theta}{\partial x_2}(x_1\,,x_2)\quad , \qquad \sigma_{23} = -\frac{\partial\theta}{\partial x_1}(x_1\,,x_2) \tag{3.16}$$

The tangential derivative of θ along the boundary of the cross section Σ amounts to :

$$\frac{\partial\theta}{\partial\tau} = -\frac{\partial\theta}{\partial x_1} n_2 + \frac{\partial\theta}{\partial x_2} n_1 = \sigma_{13}\, n_1 + \sigma_{23}\, n_2 \quad \text{on} \quad \partial\Sigma\,. \tag{3.17}$$

According to (3.15) the boundary $\partial\Sigma$, is free of stress.

Therefore, recalling that the cross section is simply connected, we get

$$\theta = \text{constant on} \quad \partial\Sigma$$

θ being defined up to an additive constant, we can choose the constant on $\partial\Sigma$ to be 0 . Therefore if σ is in $L^2(\Omega,E)$, θ belongs to $H^1_o(\Sigma)$.

B is the ball of center 0 and radius k_o. The locking limit analysis problem can be explicitely solved :

$$\int_\Omega \pi_B(\sigma)\,dx = \int_\Omega k_o\,|\sigma|\,dx = \int_\Sigma k_o\,|\nabla\theta|\,dx_1\,dx_2\ ,$$

$$\int_{\partial\Omega} \sigma.n\; u_o\,ds = \int_{x_3=0} 0 + \int_{x_3=h} (-x_2\,\sigma_{13} + x_1\,\sigma_{23})\,ds$$

$$= -\int_\Sigma \left(\frac{\partial\theta}{\partial x_2}x_2 + \frac{\partial\theta}{\partial x_1}x_1\right)dx_1\,dx_2 = \int_\Sigma \theta\,dx_1\,dx_2\,.$$

The locking limit analysis problem amounts to

$$\bar\lambda_\ell = \underset{\substack{\theta\in H^1_o(\Sigma)\\ \int_\Sigma \theta\,dx=1}}{\text{Min}} \int_\Sigma k_o\,|\nabla\theta|\,dx \tag{3.18}$$

This minimization problem has been already encountered by STRANG [18] in the determination of the limit load of a vertical column submitted to body forces (anti-shear problem). The following conclusions of STRANG are especially meaningful in the example here considered :

a) The problem (3.18) *does not admit* a solution θ in $H^1_o(\Sigma)$, nor in $W^{1,1}_o(\Sigma)$. The proper space to work with, is $BV(\Sigma)$; the stress tensor σ is therefore in $M^1(\Sigma)$ by virtue of (3.16). In $BV(\Sigma)$ *the solutions* θ *of* (3.18) *are characteristic functions of sets* with rather smooth boundaries. According to (3.16), the stress tensor is a *Dirac distribution* on the boundaries of these sets : these lines will be called *locking lines*.

b) locking lines necessarily intersect the boundary $\partial\Sigma$ of the cross-section. Therefore the stress function θ is no more constant on $\partial\Sigma$ since it jumps from 0 to 1. As a consequence its tangential derivative given by (3.17) contains Dirac distributions on $\partial\Sigma$, and in particular does not vanish in $\mathcal{D}'(\Sigma)$. *The condition of free edge* (3.15) *is not satisfied* in a distributional sense. In a more general context the boundary conditions of imposed forces have to be relaxed (cf. [16]).

c) in case of a square cross section, STRANG found out the explicit solution of (3.18). Locking lines are plotted on figure 6. Jumps of θ on $\partial\Sigma$ are noticeable. The curved parts of locking lines are circle portions.

3.4. Possible extensions.

A theory of locking of *structures* can be proposed. Strains are taken in a generalized sense : rotations, deflections, angles... In the example of a robot, the limit strain ε_o can model the free motion of the joints. The problem is to determine which joints will be locked under a specified loading and which imposed displacements are admissible. We expect the stresses to concentrate on the locked joints which therefore require an adequate reinforcement.

4. CONCLUSIONS

A possible modelling of hysteresis phenomena, accounting for locking effects has been proposed. The case of ideal locking materials has been considered and a special emphasis has been set on the locking limit analysis.

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