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Friction and homogenization of a boundary

1° INTRODUCTION

The equilibrium problem for an elastic body on a rigid support with dry friction (Coulomb's law) seems to be an open free boundary problem. The main difficulty is the lack of a variational principle associated to the problem and the consequent failure of the convex analysis technique. In fact, with standard notations (which will be given in the sequel) the *formal* variational formulation of the problem is (cf. [1] or [2] sect. 5.4.4)

$$\left.\begin{aligned} & u \in K \\ & \int_\Omega a_{ijkh}\, e_{kh}(u)\, e_{ij}(v-u)\, dx + \int_{\Gamma_2} k|\sigma_N|(|v_T| - |u_T|)\, ds \geq \\ & \qquad \geq \int_\Omega f(v-u)\, dx \quad \forall v \in K \\ & K = \{v \mid v_N \leq 0 \text{ on } \Gamma_2\} \end{aligned}\right\} \qquad (1.1)$$

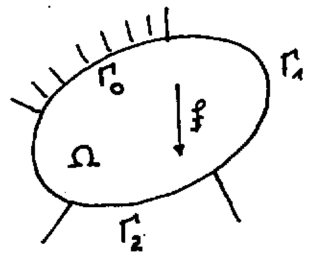
The term containing $|\sigma_N|$ is not defined for $\sigma \,.\, n \in H^{-1/2}(\Gamma_2)^3$ and is not the subgradient of a functional. Several mathematical attempts have been made in order to overcome this difficulty: non local friction [1], fixed point techniques (quasi-variational inequalities [3][4]). The former introduces a modification of the law while the latter involves a relation between the friction coefficient and the elasticity coefficients.

In fact, friction seems to be a surface phenomenon associated with roughness. In the present work we apply the *homogenization of boundaries* to the classical (without friction) Signorini's problem on a boundary having small undulations. A small parameter ε is associated with the size of the corrugations. In fact the limit problem (homogenized) is not a dry friction problem. It is a new well posed (variational) problem for which the stress vector on the boundary is contained in the conjugate cone (instead of a halfspace). This law was already proposed in [5].

In fact our result is not very surprising for two reasons. First the hypothesis of small displacements is not probably fitted for the physical problem. Second the Signorini's problem is of standard type (minimization of some energy) and this property is preserved by homogenization of the boundary. As a result, our study is an example of homogenization of a boundary, but it does not furnish a justification of the Coulomb's dry friction law. This justification has to be done.

2° SETTING OF THE PROBLEM

The classical Signorini's problem (without friction) is the following (see for instance (2)). Let Ω be a bounded connected problem in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ formed by three disjoint surfaces $\Gamma_0, \Gamma_1, \Gamma_2$. The solid body fills Ω, is clamped on Γ_0 and free on Γ_1. The surface Γ_2 is such that the body may either lie or part on a rigid support.



$$\left.\begin{array}{l} \dfrac{\partial\sigma_{ij}}{\partial xj} + f_i = 0 \text{ in } \Omega \\[2ex] u_i = 0 \text{ or } \Gamma_0,\ \sigma_{ij}n_j = 0 \text{ on } \Gamma_1 \end{array}\right\} \tag{2.1}$$

$$u_N \leq 0,\ \sigma_N \leq 0,\ \sigma_T = 0,\ u_N\sigma_N = 0 \text{ on } \Gamma_2 \tag{2.2}$$

$$\sigma_{ij} = a_{ijkh}\, e_{khx}\,(u),\ e_{khx}\,(u) = 1/2\ \left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right)$$

Classical notations are used: in particular a_{ijkh} are the elastic coefficients. We consider them to be constant and satisfying the standard conditions of symmetry and ellipticity.

The variational formulation of (2.1)-(2.3) is as follows. We define the Hilbert space V, the closed convex set K and the bilinear and linear forms a and L by

$$V = \{v | v = (v_i),\ v_i \in H^1(\Omega),\ v_i = 0 \text{ on } \Gamma_0\ i = 1, 2, 3\}$$

$$K = \{v | v \in V,\ v_N \le 0 \text{ on } \Gamma_2\}$$

$$a(u,v) = \int_\Omega a_{ijkh}\, e_{khx}(u)\, e_{ijx}(v)\, dx$$

$$L(v) = \int_\Omega f v dx$$

Then the problem amounts to:

$$\left.\begin{array}{l} \text{find } u \in K \text{ such that:} \\ a(u, v-u) \ge L(v-u) \quad \forall v \in K \end{array}\right\} \qquad (2.4)$$

This problem has a unique solution, it is equivalent to the minimization problem:

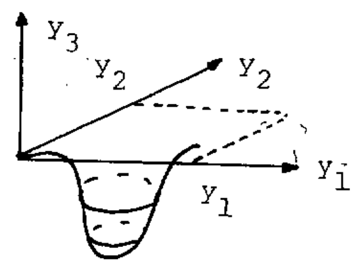
$$\min_{v \in K} \phi(v) = 1/2\, a(v,v) - L(v)$$

We now consider a special case of this situation for domains $\Omega = \Omega_\varepsilon$ depending a parameter $\varepsilon \to 0$ as follows. We consider the plane (Oy_1y_2) shared into periods $Y =]0,Y_1[\times]0,Y_2[$. We give a positive Y-periodic function $F(y_1y_2)$ of class C^∞ taking value zero in a neighbourhood of the boundary of the period. We then consider the surface Σ defined by:

$$y_3 = -F(y_1, y_2)$$

and let Σ_ε be its homothetic with ratio ε.

$$x_3 = -\varepsilon F\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$$



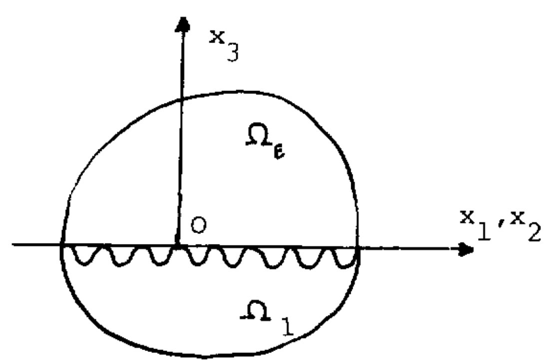
We consider an open connected domain Ω_1 having a non empty intersection with the plane $x_3 = 0$. The domain Ω_ε and the limit domain Ω_0 are defined by

$$\Omega_\varepsilon = \Omega_1 \cap \{x | x_3 > -\varepsilon F(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})\}$$

$$\Omega_0 = \Omega_1 \cap \{u | x_3 > 0\}$$

The undulated boundary is:

$$\Gamma_2^\varepsilon = \Omega_1 \cap \Sigma_\varepsilon$$



The corresponding Signorini's problem amounts to search for $u^\varepsilon \in V^\varepsilon$ satisfying the analogous of (2.4) with:

$$V^\varepsilon = \{v \mid v = (v_i),\ v_i \in H^1(\Omega^\varepsilon),\ v_i = 0 \text{ on } \Gamma_0\}$$

$$K^\varepsilon = \{v \mid v \in V^\varepsilon,\ v.n \le 0 \text{ on } \Gamma_2^\varepsilon\}$$

3° ASYMPTOTIC EXPANSION AND CONSEQUENCES

Following the classical process of boundary homogenization ([6] sect. 5.7) we define the domain B.

$$B = \{y \mid y_i \in]0,Y_i[,\ i = 1,2,\quad y_3 > -F(y_1,y_2)\}$$

Then we expand the stress and displacement fields:

$$\sigma^\varepsilon(x) = \sigma^0(x,y) + \varepsilon\sigma^1(x,y) + \dots + \varepsilon^i\sigma_i(x,y) + \dots \qquad y = \frac{x}{\varepsilon} \tag{3.1}$$

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x,y) + \dots + \varepsilon^i u_i(x,y) + \dots \qquad y = \frac{x}{\varepsilon} \tag{3.2}$$

with σ^i, u^i B-periodic. Moreover u^1 must satisfy the boundary layer condition:

$$\lim_{y\to+\infty} \operatorname{grad}_y u^1 = 0 \iff \lim_{y\to+\infty} e_y(u^1) = 0 \tag{3.3}$$

The expansions of (2.1), (2.3) give at order ε^{-1} and ε^0:

$$\left.\begin{aligned} &\frac{\partial\sigma_{ij}}{\partial y_j} = 0 \quad \text{in } B \\ &\sigma_{ij} = a_{ijkh}\left[e_{khx}(u^0) + e_{khy}(u^1)\right] \quad \text{in } B \end{aligned}\right\} \tag{3.4}$$

$$\frac{\partial\sigma_{ij}}{\partial x_j} + f_i = 0 \text{ in } \Omega^0 \tag{3.5}$$

in order to expand the boundary condition (2.2) we define two conjugate convex cones of $\mathbb{R}^3$:

$$\Gamma = \{\beta | \beta.n(y) \leq 0 \quad \forall y \in \Sigma\}$$

$$\Gamma^* = \{\tau | (\tau,\beta) \leq 0 \quad \forall \beta \in \Gamma\}$$

(2.2) at order ε^0 gives

$$u^0 \in \Gamma. \tag{3.6}$$

The tangential components of $\sigma.n$ are zero and

$$u^0(x)n(y) < 0 \implies \sigma_{ij}(x,y)n_i(y)n_j(y) = 0 \tag{3.7}$$

$$u^0(x)n(y) = 0 \implies \sigma_{ij}(x,y)n_i(y)n_j(y) \leq 0 \tag{3.8}$$

At order ^{1}, if $\underline{u^0 \in \mathrm{Int}\Gamma}$ (interior of Γ) we are in the situation (3.7) and (2.2) gives no new condition on u^1. On the other hand, if $u^0 \in b\Gamma$ (boundary of Γ) there is a subset of Σ, denoted Σ' where (3.8) holds and we have:

$$u^1(x,y)n(y) \leq 0 \qquad \forall y \in \Sigma' \tag{3.9}$$

For sake of simplicity we admit that Σ' is either Σ(if $u^0 = 0$) or a set with zero measure. This happens in particular if ε does not contain any plane portion (apart from $y_3 = 0$).

Thus, the so called *local problem* in B (cf. [6]) (x is a parameter) amounts to find a B-periodic function $u^1(y)$ satisfying (3.3)(3.4)(3.6)(3.7)(3.8)(3.9) where u^0 is given (in fact $u^0 \in \mathbb{R}^3$ and $e_{ijx}(u^0) \in \mathbb{R}^6$ are given). We shall see later that the local problem has a solution if u satisfies some compatibility conditions. These conditions (see later) constitute boundary conditions to be satisfied by $u^0(x)$ on $\{x_3 = 0\}$ in order to define a boundary value problem for u^0 in Ω_0. This limit problem will be explicitly given in Section 5.

In order to obtain the compatibility conditions we multiply (3.4) by any

$$0 = \int_B \frac{\partial \sigma_{ij}}{\partial y_j} \beta i \, dy = \int_{\partial B} \sigma^0{}_{ij} n_j \beta_i ds$$
$$= a_{ijkh}\, e_{khx}\, (u^0)\, \beta_i \int_{\partial B} n_j ds + \int_{\partial B} a_{ijkh}\, e_{khy}\, (u^1) n_j \beta_i ds \tag{3.10}$$

But $\int_{\partial B} n_j ds = 0$ and $\int_{\Sigma} n_j ds = - \delta_j Y_1 Y_2$

bearing in mind the periodicity of u^1, (3.10) gives:

$$a_{i3kh}\, e_{khx}\, (u^O) \mid x_3 = 0 \quad \beta_i = \int_{\Sigma} \sigma_N^O \beta_N ds \tag{3.11}$$

and by means of the notation:

$$\sigma_{ij}(u^O) = a_{ijkh}\, e_{khx}(u^O)$$

(3.11) becomes, by virtue of (3.6)(3.7)(3.8)

$$\sigma_{i3}(u^O)\beta_i \geq 0 \quad \forall \beta \in \Gamma \quad \sigma_{i3}(u^O)\, u^O_i = 0 \text{ on } x_3 = 0 \tag{3.12}$$

standard properties of convex analysis show that this is equivalent to either (3.13) or (3.14) where I_Γ denotes the indicative function of Γ and ∂I_Γ its subdifferential (see [7]):

$$\left.\begin{array}{l} u^O \in \Gamma \\ - \sigma(u^O).n \ \in \partial I_\Gamma(u^O) \end{array}\right\} \quad \text{on } x_3 = 0 \tag{3.13}$$

$$\left.\begin{array}{l} - \sigma(u^O).n \ x_3 = 0 \in \Gamma^* \\ u^O \in \partial I_{\Gamma^*}(- \sigma(u^O).n) \end{array}\right\} \quad \text{on } x_3 = 0 \tag{3.14}$$

4° INDICATIONS ABOUT EXISTENCE AND UNIQUESNESS OF THE LOCAL PROBLEM

We sum up the local problem. Let $u^O \in \mathbb{R}^3$, $e_{ijx}(u^O) \in \mathbb{R}^6$ be given, satisfying the compatibility conditions ((3.12),(3.13) or (3.14)): find a B-periodic vector u^1 satisfying:

$$\left.\begin{array}{l} \dfrac{\partial \sigma^o_{ij}}{\sigma Y_j} = 0,\ \sigma^o_{ij} = a_{ijkh}\,[e_{khx}\,(u^O) + e_{ijy}(u^1)] \quad \text{in } B \\ \lim\limits_{Y_z \to +\infty} e_y\,(u^1) = 0, \text{ tangential components of } \sigma_{ij}n_j \text{ zero on} \\ u^1 n(y) \leq 0 \text{ and } \sigma_{ij}n_j n_i \leq 0 \quad \forall y \in \Sigma',\ \sigma_{ij}n_j n_i = 0 \ \forall y \in \Sigma - \Sigma' \end{array}\right\} \tag{4.1}$$

In order to give a variational formulation of this problem, we define a space and a convex set K as follows. Let B_R be the domain defined by: $B_R = \{y \in B,\ y_3 < R\}$. Let $\mathcal{E}$ be the set of the B-periodic vector functions of class C^∞ which are constant for sufficiently large y_3. Then V is the completed space of $\mathcal{E}$ for the norm associated with the scalar product:

$$(u,v) = \int_B e_{ijy}(u)\, e_{ijy}(v)\, dx + \int_{B_R} u_i v_i dy$$

$$K = \{v | v \in V \quad v.n|_{\Sigma'} \leq 0\} \tag{4.2}$$

The problem (4.1) is then equivalent to the following variational problem:

Find $u^1 \in K$ such that $\forall v \in K$

$$\int_B a_{ijkh}\, e_{khy}(u^1)\, e_{ijy}(v-u^1)\, dy + \int_\Sigma a_{ijkh}\, e_{khx}(u^0)\big|_{x_3=0}\, n_j (v_i - u_i^1)\, ds \geq 0 \tag{4.3}$$

or equivalently the minimization on K of the functional:

$$\Phi(v) = 1/2 \int_B a_{ijkh}\, e_{khy}(v)\, e_{ijy}(v)\, dy + \int_\Sigma a_{ijkh}\, e_{khx}(u^0)\Big|_{x_3=0} n_j v_i ds \tag{4.4}$$

It is to be noticed that in the case $u^0 \in b\Gamma$, $u^0 \neq 0$, which is a very special case, Σ' is a part of Σ with zero measure. Thus it is not obvious that K is closed for the strong topology of V (and it is probably not: think to the dense embedding of $H^{1/2}_{00}$ into $H^{1/2}$ [8]). Physically the small deformations hypothesis is probably violated and the problem should be formulated in another framework. This point deserves a deeper study.

<u>Case $u^0 = 0$:</u> In this case $\Sigma' = \Sigma$. We admit that the compatibility condition satisfied in such a way that:

$$\sigma_{i3}(u^0) \in \text{Int}\ \Gamma \quad \text{on } x_3 = 0 \tag{4.5}$$

Then a solution u^1 exists because:

$$\lim_{\substack{v \to +\infty \\ v \in K}} \Phi(v) = +\infty \tag{4.6}$$

Case $u^0 \in \mathrm{Int}\Gamma$: $u^1 = 0$ is a solution of the problem.

Case $u^0 \in b\Gamma$, $u^0 \neq 0$: We admit that the compatibility condition is satisfied in such a way that:

$$\left.\begin{array}{l} \sigma_{i3}(u^0)|_{x_3=0} \ \gamma_i = 0 \\ \\ \gamma_i n_i(y) \leq 0 \quad \forall y \in \Sigma' \end{array}\right\} \Rightarrow \exists \lambda \ \ \gamma = \lambda u^0$$

In this case we define a space $\dot{V}$ as the quotient space of V by the straight line $\{\lambda u^0, \lambda \in \mathbb{R}\}$. We note that $\Phi(v)$ take the same for all the elements of an equivalence class and consequently is a functional $\dot{\Phi}$ on $\dot{V}$. The same thing holds for K from which we get a convex set $\dot{K}$ of $\dot{V}$. The existence of a solution then follows from a property analogous to (4.6) in $\dot{V}$, $\dot{K}$.

5° HOMOGENIZED BOUNDARY CONDITION AND COMPLEMENTS

5.1. The limit problem

According to the considerations of Section 3, the homogenized problem for u^0 in Ω_0 is:

$$\frac{\partial \sigma_{ij}(u^0)}{\partial x_j} + f_i = 0, \ \sigma_{ij} = a_{ijkh} \ e_{khx}(u^0) \ \text{in } \Omega_0 \tag{5.1}$$

$$u^0 = 0 \text{ on } \Gamma_0, \ \sigma_{ij}(u^0) n_j = 0 \text{ on } \Gamma_1 \tag{5.2}$$

$$u^0 \in \Gamma, \quad -\sigma(u^0).n \in \partial I_\Gamma(u^0) \ \text{on } \partial\Omega_0 \cap \{x_3 = 0\} \tag{5.3}$$

This problem has one and only one solution. Indeed, if K^0 is defined by (5.4) the problem (5.1)-(5.3) is equivalent to (5.5).

$$K^0 = \{v | v \in (H^1(\Omega^0))^3, \ u|_{x_3=0} \in \Gamma, \ u = 0 \text{ on } \Gamma_0\} \tag{5.4}$$

$$\text{Find } u^0 \in K^0 \text{ such that } \ \forall v \in K^0 \tag{5.5}$$

$$\int_\Omega a_{ijkh} \ e_{khx}(u^0) \ e_{ijx}(v-u^0)\,dx \geq \int_\Omega f_i \ (v_i - u_i^0)\,dx$$

5.2. On the structure of the friction laws [9]/[10]:

The main difficulty of the Coulomb's law is that it is a *non standard* one (this will be precised later on) and therefore cannot be handled by classical reasonings of Convex Analysis.

We note that for an elastic body the displacement field on Γ_2 is a *global state variable*. Indeed if this displacement field, now denoted α, is known the displacement in the whole body is given by the variational principle:

$$W(\alpha) = \mathrm{Min} \qquad W(u) = 1/2\int_\Omega a_{ijkh}\, e_{khx}(u)\, e_{ijx}(u)\,du - \int_\Omega fudx \tag{5.6}$$

$$u = \alpha \text{ on } \Gamma_2$$

$$u = 0 \text{ on } \Gamma_0$$

The thermodynamical force A associated with α is:

$$A = -\frac{\partial W}{\partial \alpha} = -\sigma.n \text{ on } \Gamma_2 \ .$$

The two laws of thermodynamics show that the dissipated power is:

$$\mathcal{D} = \int_{\Gamma_2} A\dot{\alpha} = -\int_{\Gamma_2} \sigma.n\ \dot{u}ds \geq 0 \ . \tag{5.7}$$

We shall say that the dissipative process is *standard* if there exists a convex, l.s.c. function φ such that:

$$\dot{\alpha} = \frac{\partial \varphi}{\partial A}(A) \qquad \varphi(A) \geq \varphi(0) = 0 \tag{5.8}$$

The friction law (5.8) is an evolution law which accounts for time effects. Dry (or static) friction laws are built on the same model:

$$\alpha = \frac{\partial \varphi}{\partial A}(A) \tag{5.9}$$

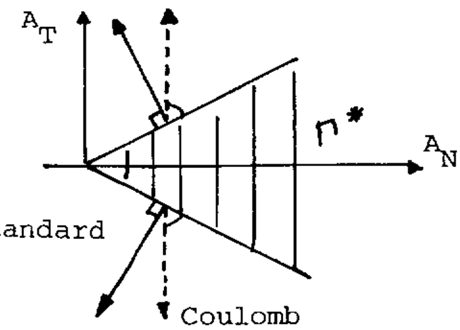
Examples

1. Viscous friction

$$\varphi(A) = 1/2\ k|_T|^2 \qquad \dot{u}_T = -k\sigma_T,\ \dot{u}_N = 0 \quad \text{(evolution)}$$

or $u_T = - k\sigma_T$, $u_N = 0$ (dry) ..

2. $\varphi(A) = I_{\Gamma^*}(A)$ where $\Gamma^* = \{A \mid A_N \geq 0,\ |A_T| \leq k|A_N|\}$



The dry friction law (5.9) is: $- \sigma.n \in \Gamma^*$, $u \in \partial I_\Gamma(-\sigma,n)$ on Γ_2 which is exactly the law (5.3) in its form (3.14).

As it can be seen on the figure the Coulomb's law is *not standard*.

6° CONCLUSIONS

The present work illustrates the technique of boundary homogenization. It proposes a law of friction where sliding is allowed only after separation. This law is a standard one and differs from the Coulomb's law. Coulomb's law is still to be justified by more accurate models.

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