

# Représentation port-Hamiltonienne de Systèmes Dynamiques non linéaires

## Application à l'appareil vocal

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17 octobre 2019

## Contexte et motivations

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## Un peu de contexte...

Mon parcours de 4 lignes :

- 2005-09 Doctorat au LMA sur les transitoires des instruments de musique à anche simple
- 2010-11 Post-doc au LFD (Buenos Aires) sur l'écoulement laryngé
- 2012-14 Post-doc au GIPSA-lab (Grenoble) sur la caractérisation de maquettes de plis vocaux
- 2014- Chargé de Recherche CNRS au LMA - équipes *Matériaux et Structures* puis *Sons* (2019-)

### Oscillations induites par écoulement :

Couplage fluide de résonateurs acoustiques et/ou mécaniques

- vibrations de structures dans le vent,
- bruit généré dans les conduites hydrauliques,
- contraction de vaisseaux sanguins,
- instruments de musique à vent
- ronflement et voix

# Une communauté large autour de la voix et de la parole

## Les usagers de la voix

- nous tous! (ou presque)  
C'est le moyen de communication principal des êtres humains!
- les usages professionnels : par ex., artistes, pédagogues

## Les thérapeutes

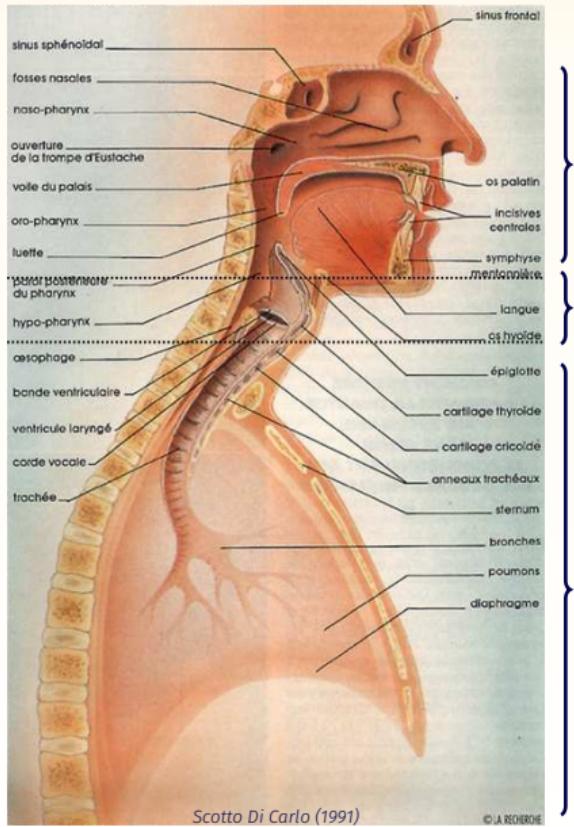
- les phoniatres,  
plus largement la médecine ORL
- les orthophonistes  
prévention et rééducation des troubles organiques et du développement
- les psychanalystes

## Au sein de la recherche :

De nombreuses disciplines impliquées :

- la physique de la production de la voix et de la parole
- la phonétique
- la phonologie et la linguistique
- la dialectique
- la psychoacoustique et la perception de la parole
- les aspects cognitifs de la communication parlée  
mécanismes d'encodage et de décodage, contrôle neuromoteur
- le traitement automatique de la parole : reconnaissance, transformation, synthèse

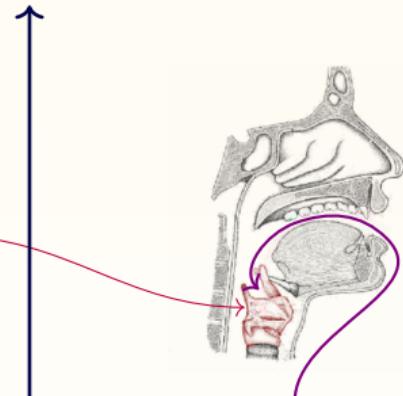
# Anatomie de l'appareil vocal humain



**Étage articulatoire :**  
les voies aériennes supérieures  
résonateurs acoustiques

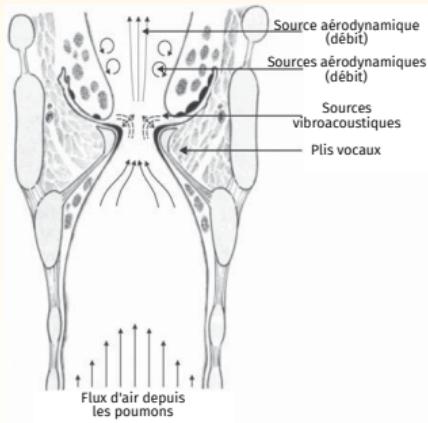
**Étage phonatoire : le larynx**  
conversion en énergie vibratoire  
(mécanique et acoustique)

**Étage respiratoire : la soufflerie**  
source d'énergie aérodynamique



Plis vocaux  
(= cordes vocales)

# Principe physique : une interaction fluide-structure-acoustique

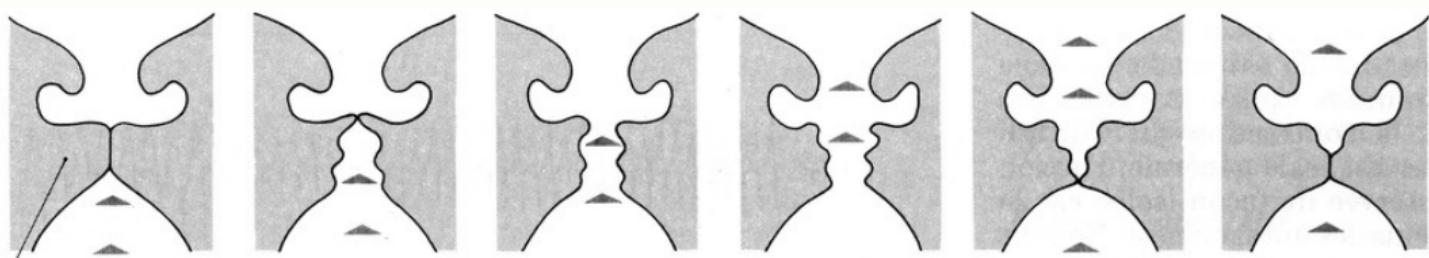


Compétition entre :

- les forces fluides liées à la distribution de pression dans le canal
- les forces de rappel dues à l'élasticité des tissus

L'équilibre devient instable et des **auto-oscillations** apparaissent :

- les **vibrations des plis vocaux**...
- modulent le débit d'air entrant dans le conduit vocal...
- qui est alors le siège d'**ondes acoustique**
- qui rétroagissent sur les plis et l'écoulement.



Théorie aéro-myoélastique

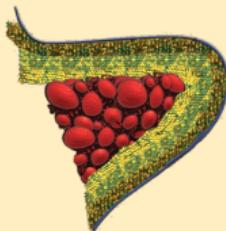
(van den Berg, 1957)

L'auto-oscillation est principalement contrôlée par les propriétés dynamiques des plis

# Un principe simple, une réalité plus complexe...

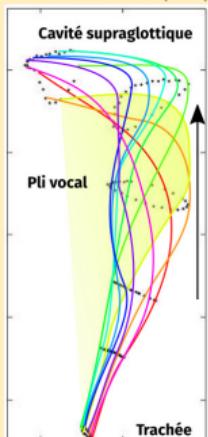
Côté solide...

Structures  
multi-couches  
hétérogènes



Vibration non uniforme  
dans le sens de l'écoulement,  
& dans la longueur des plis

Déformation des tissus  
*ex-vivo hémilarynx*  
Boessenecker (2007)



Commissure  
postérieure



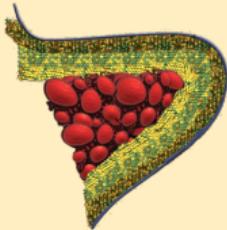
Commissure  
antérieure

←Fermeture « Zip »  
(Glottovibrogramme)  
t Karakozoglu (2012)

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Structures multi-couches hétérogènes



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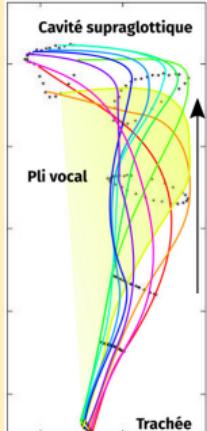
Commissure postérieure



Commissure antérieure

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## Côté acoustique...

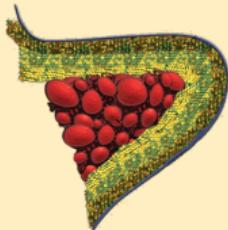
Un résonateur fortement variable au cours du temps : articulation, aéro-acoustique possible



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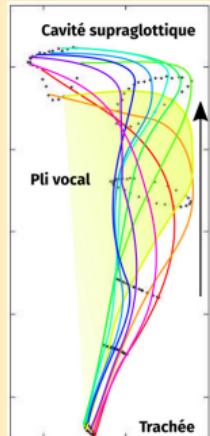
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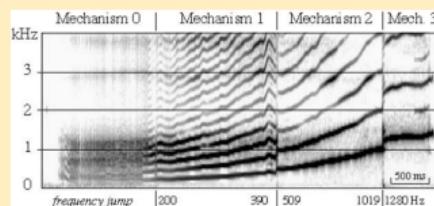
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...pour une production variée

Glissando

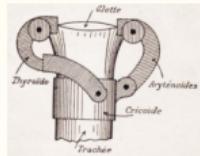
Roubeau (2009)



Bifurcations entre régimes de vibration distincts

# Côté fluide : 50 ans de travaux d'expériences sur maquettes

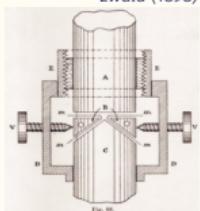
Muller (1839)



Helmholtz (1877)



Ewald (1898)

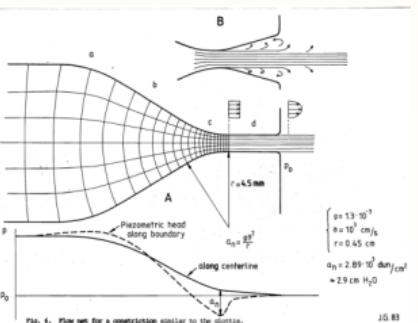
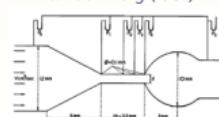


## Maquette rigide et écoulement stationnaire

Mesure distribution de pression

& résistance aérodynamique du larynx

van den Berg (1957)

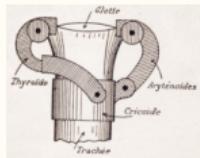


(1983)

Ananthapadmanabha

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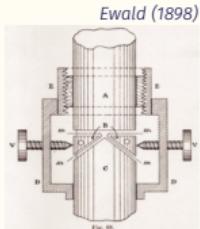
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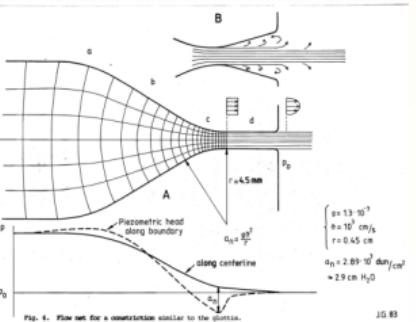
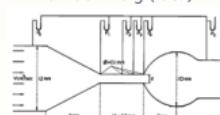


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Mesure distribution de pression

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van den Berg (1957)



## Instationnarité fluide (à géométrie fixe)

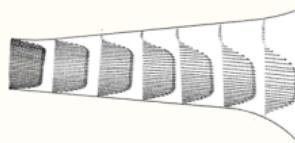
Dynamique d'établissement de l'écoulement

Mesure dans le canal

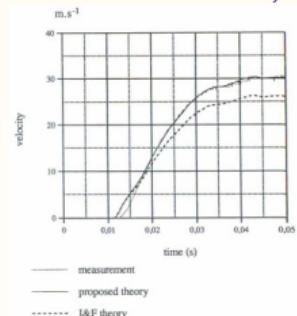
Distribution de vitesse

Effet d'asymétrie géom.

Scherrer (2001)



Transitoire  $v_{jet}$



Pelorson (1994-95)

Formation d'un jet  
dans la partie divergente



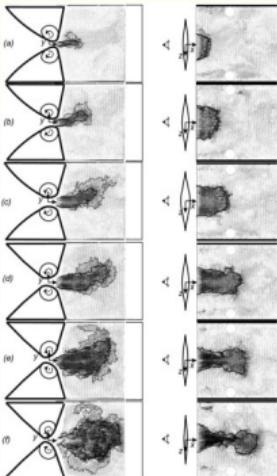
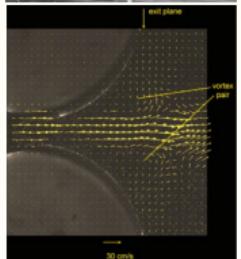
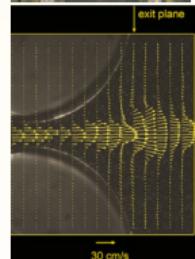
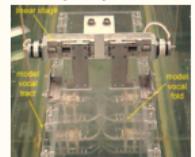
## Plus récemment avec des géométries variables...

### Maquettes motorisées (geom. forcée)

Krane (2007)

↔ eau! ↔

Triepl (2005 & 2010)



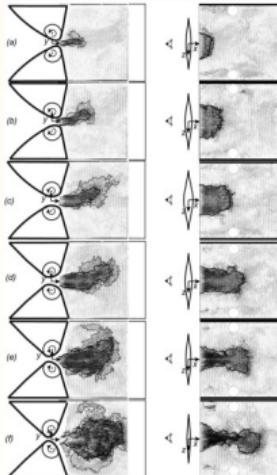
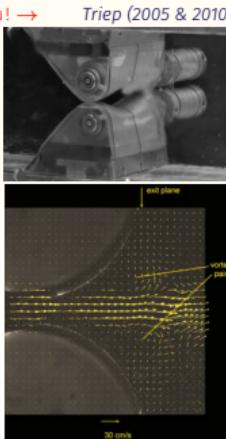
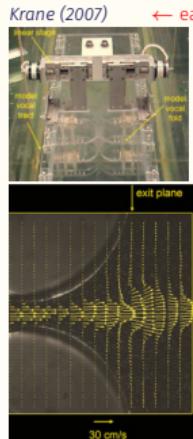
Effet du mouvement des parois non négligeable  
aux faibles ouvertures glottiques.

Point de séparation variable

Formation de dipôles de vorticité et latéralisation du jet

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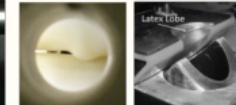


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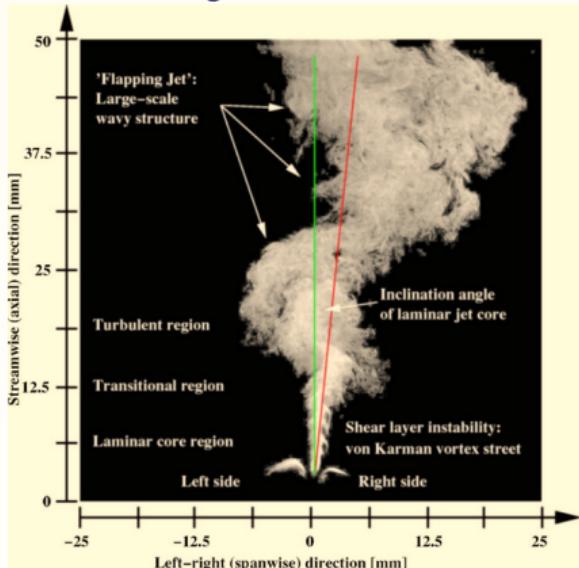
## Maquettes auto-oscillantes



- + Mesure des seuils d'oscillation
- + Effets du couplage avec le conduit vocal
- + Étude de la dynamique du jet
- + Accès aux vibrations induites par l'écoulement
- Les techniques d'imagerie du fluide deviennent inutilisables dans le canal glottique.

## Plus récemment avec des géométries variables...

Phénoménologie très riche!



Neubauer (2007)

## Maquettes auto-oscillantes



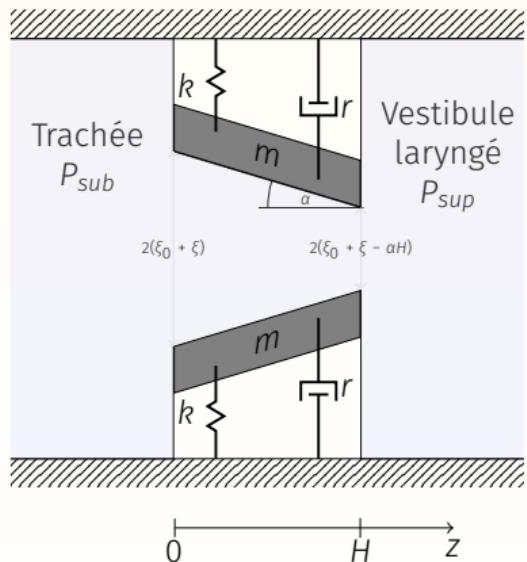
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## Une vision (personnelle) de la littérature

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# Auto-oscillations d'un oscillateur élémentaire

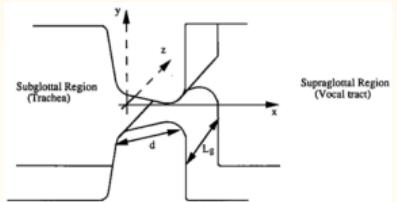
Quels sont les ingrédients nécessaires pour qu'un résonateur simple entre en vibration sous l'effet d'une pression subglottique constante ?



Hypothèses :

- Plis : oscillateurs élémentaires
- Vibration transverse des plis
- Géométrie simpliste : canal glottique en forme de trapèze  $a(z) = 2L_g (\xi_0 + \xi - \alpha z)$
- Forces de pression dans le canal glottique uniquement

# Grandeurs caractéristiques et modélisation



Longueur glotte	$L_g \sim 14 - 18\text{mm}$
Épaisseur plis	$d \sim 5 - 9\text{mm}$
Largeur glotte	$h \sim 0 - 3\text{mm}$
Pression transglottique	$\Delta P_{tot} \sim 0.4 - 2\text{kPa}$
Vitesse dans la glotte	$u_0 \sim 10 - 40\text{m/s}$
Fréquence fondamentale	$f_0 \sim 100 - 200\text{Hz}$
Longueur d'onde acoustique	$\lambda \sim 1.7 - 3.4\text{m.}$

## Nombres adimensionnés

Aspect ratio  $h/L_g \ll 1$  pb 2D

Nb Mach  $u_0/c \ll 1$  quasi incompressible

Nb Strouhal  $f_0 d/u_0 \sim 0.01$  quasi-station.

Nb Reynolds  $hu_0/\nu \sim 10^3$   
couche limite visqueuse

## Modèle le plus couramment utilisé

Ishizaka (1972)

- Écoulement incompressible : débit  $U(t) = a(x, t)v(x, t) = cst$
- En entrée  $P_{sub} = P(0) + \frac{1}{2}k_c\rho \frac{U^2}{a(0)^2}$   
Coefficient de contraction  $k_c \simeq 1.1$
- En sortie, jet et dissipation de l'énergie cinétique par « turbulence »  
$$P_{sup} = P(H) + \frac{1}{2}k_e\rho \left(\frac{U}{a(H)}\right)^2$$
  
Faible recouvrement de pression  $k_e \simeq 0.2$

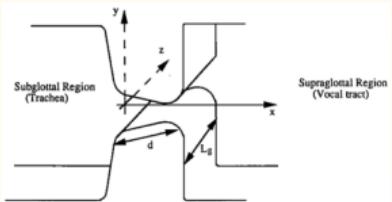
## Distribution de pression dans le canal

Analogie th. Bernoulli :

$$P(x, t) = P_{sub} - \frac{1}{2}k_c\rho \frac{U^2}{a(x, t)^2}$$

à relier à la pression en entrée du conduit vocal  
 $P(H, t) = P_{supra}(t) = p_{acous}(t).$

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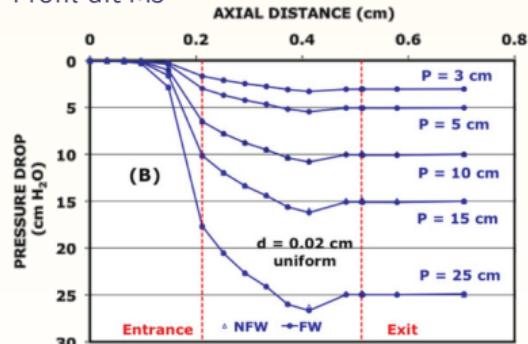
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# Grandeurs caractéristiques et modélisation

En accord avec mesures sur maquette *rigide*  
Profil dit M5



Fulcher (2011)

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# Analyse linéaire de stabilité

Dynamique pour le déplacement de chaque pli :

$$m\ddot{\xi}(t) + r\dot{\xi}(t) + k\xi(t) = F_p(P_{sub}, P_{sup}, \xi, \alpha)$$

avec  $F_p$  résultante des forces de pression

Paramètre de contrôle principal :  $P_{sub}$

Démarche classique :

1. Recherche de l'état d'équilibre  $\xi^*$  en fonction de  $P_{sub}$

$$\xi^* = f^*(P_{sub}, \alpha, k)$$

Rem : Canal droit  $\alpha = 0 \Rightarrow \xi^* = 0$

2. Stabilité de petites perturbations  $\hat{\xi}$  autour de l'équilibre  $\xi^*$

$$m\ddot{\hat{\xi}} + r\dot{\hat{\xi}} + k(\xi^* + \hat{\xi}) = F_p(P_{sub}, P_{sup}, \xi^* + \hat{\xi}, \alpha)$$

**Modèle 0 : sans résonateur acoustique ( $P_{sup} = 0$ )**

La base de la théorie Source-filtre (Fant, 1960)!

Linéarisation de la résultante des forces de pression :

$$F_p(P_{sub}, 0, \xi^* + \hat{\xi}, \alpha) \sim F_p(P_{sub}, 0, \xi^*, \alpha) + \frac{\partial F_p}{\partial \xi} \hat{\xi}$$

soit

$$m\ddot{\hat{\xi}}(t) + r\dot{\hat{\xi}}(t) + \left(k - \frac{\partial F_p}{\partial \xi}\right) \hat{\xi}(t) = 0$$

Sa fréquence de résonance varie avec  $P_{sub}$ ...  
...mais l'oscillateur reste amorti

**Pas d'auto-oscillation possible!**

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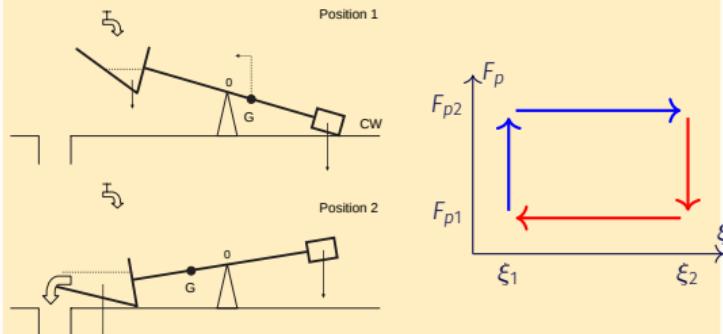
Il faut réussir à **modifier l'amortissement apparent**, avec des forces de pression glottique  $F_p$  agissant **en déphasage** avec le mouvement des plis.

## Oscillateur à relaxation

Garrel (2008)

Analogie du vase à bascule :

- Phase de **stockage** : remplissage jusqu'à la bascule
- Phase de **relaxation** : vidage jusqu'au retour en haut



Comportement hystérotique :

- en phase fermée, la pression subglottique s'accumule...
- jusqu'à ce que la glotte s'ouvre suffisamment...
- pour que la pression chute ...
- et que les plis reviennent vers leur position d'équilibre.

Modèle d'osc. grandes amplitudes  
~ « mvt de Helmholtz » des cordes frottées.

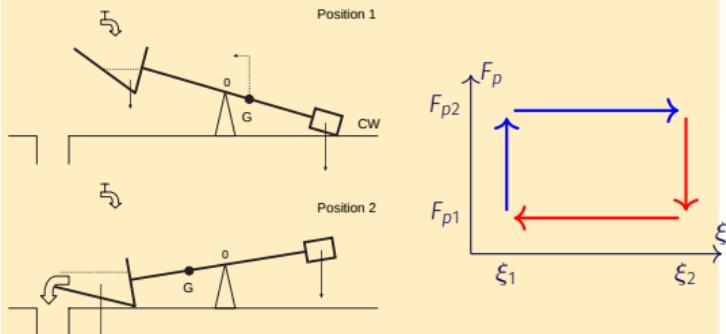
# Modèles 1, 2

## Oscillateur à relaxation

Garrel (2008)

Analogie du vase à bascule :

- Phase de **stockage** : remplissage jusqu'à la bascule
- Phase de **relaxation** : vidage jusqu'au retour en haut



Comportement hystérotique :

- en phase fermée, la pression subglottique s'accumule...
- jusqu'à ce que la glotte s'ouvre suffisamment...
- pour que la pression chute ...
- et que les plis reviennent vers leur position d'équilibre.

Modèle d'osc. grandes amplitudes  
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## Charge acoustique

Flanagan & Landgraf (1968)

Charge inertielle en-dessous des formants :

$$P_{sup}(t) = \beta \dot{U}(t) = \beta \dot{\xi}(t) \sqrt{P_{sub}}$$

⇒ Déphasage entre  $U$  et  $F_p$

$$m\ddot{\xi}(t) + \left( r - \frac{\partial F_p}{\partial P_{sup}} \beta \sqrt{P_{sub}} \right) \dot{\xi}(t) + \left( k - \frac{\partial F_p}{\partial \xi} \right) \xi(t) = 0$$

→ Amortissement apparent qui diminue avec  $P_{sub}$

→ Auto-oscillations au-delà d'une pression de seuil

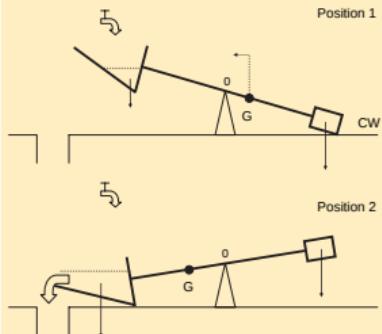
# Modèles 1, 2, 3

## Oscillateur à relaxation

Garrel (2008)

Analogie du vase à bascule :

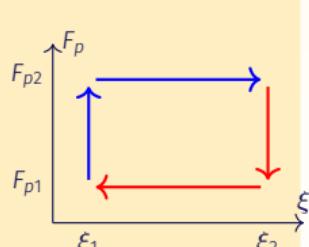
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## Onde mucosale

Titze (1988)

Retard  $\tau_m = H/c_m$  du bord amont sur le bord aval

$$\xi \left( t - \frac{z}{c_m} \right) \sim \xi(t, 0) - \dot{\xi}(t) \frac{z}{c_m} \quad \Rightarrow \quad \alpha = \frac{\dot{\xi}(t)}{c_m}$$

$$m \ddot{\xi}(t) + \left( r - \frac{\partial F_p}{\partial \alpha} \frac{1}{c_m} \right) \dot{\xi}(t) + \left( k - \frac{\partial F_p}{\partial \xi} \right) \xi(t) = 0$$

→ Amortissement diminue avec  $P_{\text{sub}}$

→ Auto-oscillations au-delà d'un seuil

$$P_{\text{onset}} = r \frac{\xi_0 c_m}{H^2 L_g} = r \frac{\xi_0}{\tau_m H L_g}$$

# Modèles 1, 2, 3, 4

## Coefficient de décharge variable

Fulchner (2006)

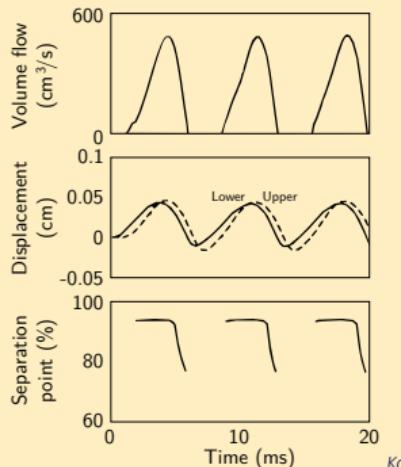
Écart au modèle d'écoulement de Bernoulli :

$$U = C_d(t) a(H) \sqrt{\frac{2(P_{sub} - P_{sup})}{\rho}}$$

Point de séparation variable

MacGowan (2010)

$$C_d = f(\xi, \dot{\xi}) \Rightarrow \frac{\partial F_p}{\partial \xi} \neq 0$$



Kaburagi (2011)

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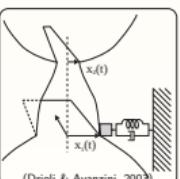
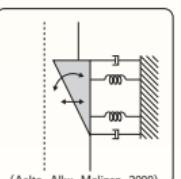
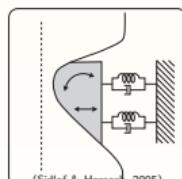
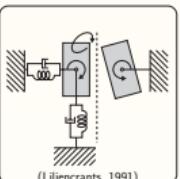
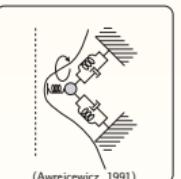
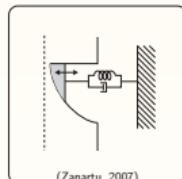
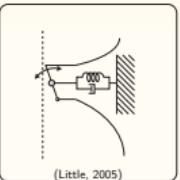
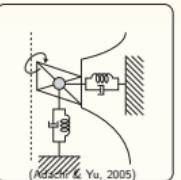
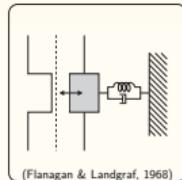
$$m \ddot{\xi}(t) + \left( r - \frac{\partial F_p}{\partial \alpha} \frac{1}{c_m} \right) \dot{\xi}(t) + \left( k - \frac{\partial F_p}{\partial \xi} \right) \xi(t) = 0$$

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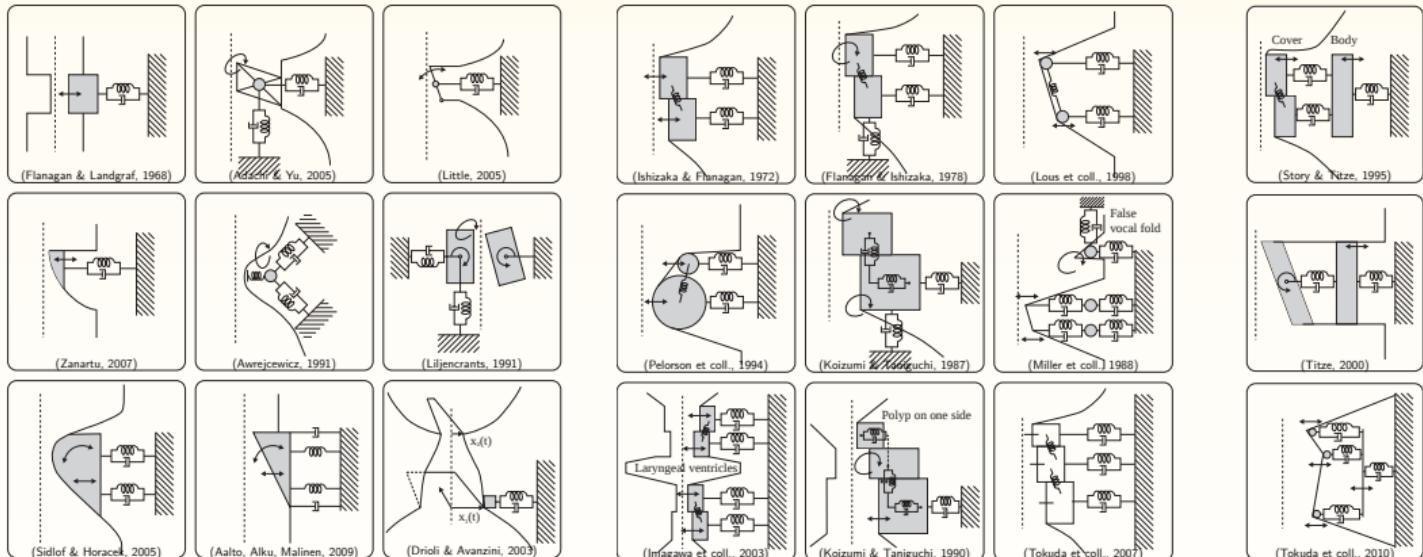
→ Auto-oscillations au-delà d'un seuil

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# Modèles 1, 2, 3, 4...N

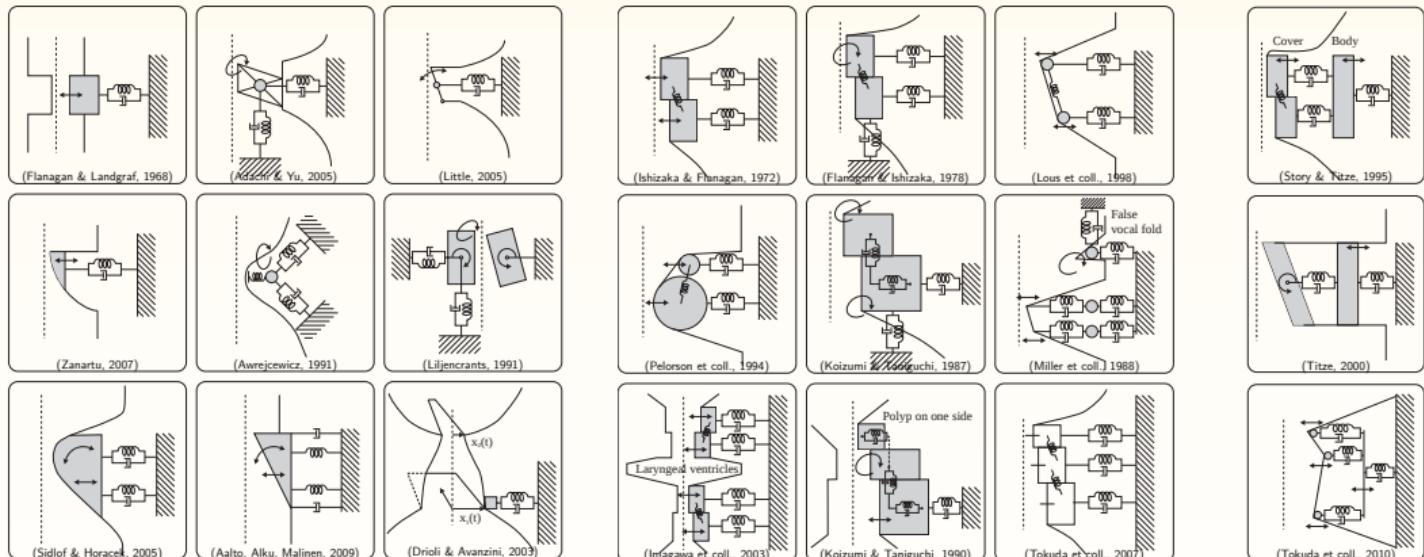


# Modèles 1, 2, 3, 4...N...N<sup>2</sup>



Birkholtz (2011)

# Modèles 1, 2, 3, 4...N...N<sup>2</sup>



Birkholtz (2011)

Une multitude de modèles avec un dénominateur commun :  
le modèle d'écoulement glottique établi pour une géométrie quasi-statique

$$\mathcal{P}_{\text{fluide} \rightarrow \text{plis}} + \mathcal{P}_{\text{plis} \rightarrow \text{fluide}} \neq 0$$

# Une interprétation historique

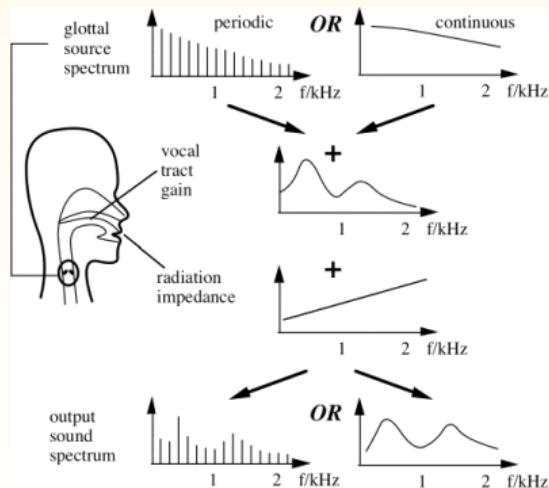
## Un ancrage dans la théorie Source-filtre de G. Fant (1960)

besoin d'expliquer la production d'un débit glottique oscillant qui puisse servir d'excitation au résonateur qu'est le conduit vocal  
→ Focus sur la composante axiale de l'écoulement glottique

Pourtant, la phonation résulte d'une instabilité fluide-structure...

## Intérêt pour la composante transverse

- Transferts de puissance entre l'écoulement et les plis
- Rôle essentiel dans l'instabilité qui aboutit à l'oscillation



## Objectif

Proposer un modèle :

**structuré** : approche par composants

**bien-posé énergétiquement** : sans énergie « tirée du chapeau »

→ Cadre théorique des Systèmes Hamiltoniens à ports

## An brief overview of the theory of port-Hamiltonian Systems (pHs)

---

# Some reminders on analytical mechanics

## Lagrangian mechanics

Principle of least action :

$$0 = \delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(t, q, \dot{q}) dt$$

$q$  : generalized coordinates

$\Rightarrow \dot{q}$  : generalized velocities

Lagrangian  $\mathcal{L}(t, q, \dot{q}) = T - V$

Euler-Lagrange equations :

$$\forall i, \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

Application to a mass  $m$  - spring  $k$  system

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2$$

$q$  = position of the mass

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$$

$\equiv$  spring's elongation

$$\frac{\partial \mathcal{L}}{\partial q} = -k q$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial q} \Rightarrow m \ddot{q} + k q = 0$$

# Some reminders on analytical mechanics

## Lagrangian mechanics

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$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$$

$$\frac{\partial \mathcal{L}}{\partial q} = -kq$$

## Hamiltonian mechanics

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} : \text{generalized momenta} \quad \Rightarrow \dot{q}(p)$$

Hamiltonian  $\equiv$  Energy (most often)

$$\mathcal{H}(t, q, p) = p \dot{q} - \mathcal{L}(t, q, \dot{q}) \text{ with } \dot{q}(p)$$

Hamilton equations :

$$\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \nabla_q \mathcal{H} \\ \nabla_p \mathcal{H} \end{pmatrix}$$

Symmetrical handling of variables  $(p, q)$   
in the phase space

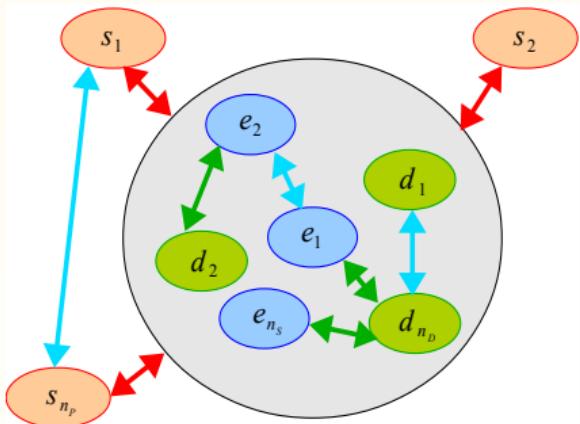
$$\mathcal{H} = \frac{1}{2} \frac{p^2}{m} + \frac{1}{2} k q^2$$

$p = m \dot{q}$  : momentum of the mass

$$\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} kq \\ p/m \end{pmatrix}$$

# Port-Hamiltonian Systems (pHs)

Extension of the Hamiltonian mechanics to handle **losses** and **external interactions**.



- Energy-storing components :

$$E = \sum_{n=1}^N e_n \geq 0$$

(energy)

- Power-dissipating components :

$$Q = \sum_{m=1}^M d_m \geq 0$$

(dissipated power)

- External components :

$$P_{\text{ext}} = \sum_{p=1}^P s_p$$

(power received by those external comp.)

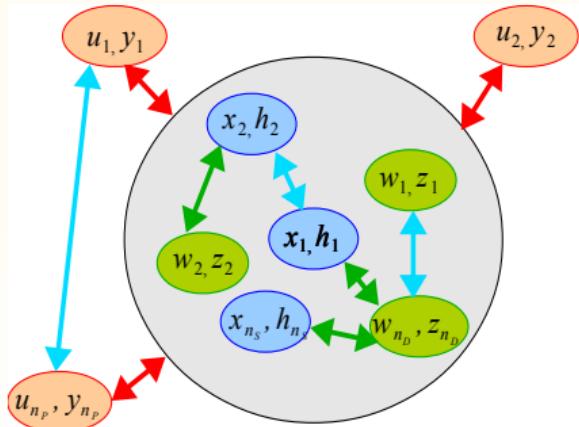
- Conservative connections :

$$\dot{E} = -Q - P_{\text{ext}}$$

(power balance)

# Port-Hamiltonian Systems (pHs)

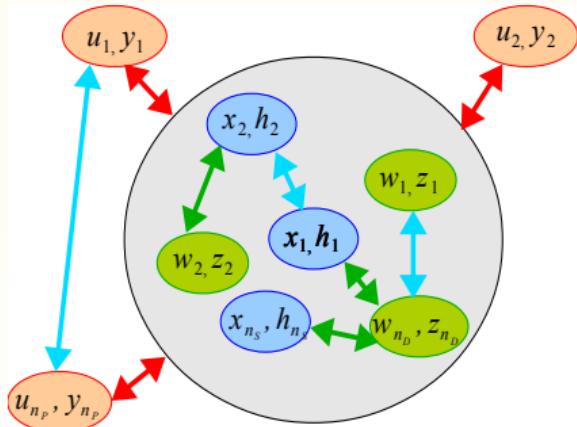
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- Power-dissipating components :  
 $Q = z(w)^T w = \sum_{m=1}^M z_m(w_m) w_m \geq 0$
- External components :  
 $P_{\text{ext}} = u^T y = \sum_{p=1}^P u_p y_p$
- Conservative connections :  
 $0 = \nabla \mathcal{H}(x)^T \dot{x} + z(w)^T w + u^T y$   
(effort  $\times$  flow : force  $\times$  velocity, tension  $\times$  current, etc...)

# Port-Hamiltonian Systems (pHS)

Extension of the Hamiltonian mechanics to handle losses and external interactions.



- Energy-storing components :  
 $E = \mathcal{H}(x) = \sum_{n=1}^N \mathcal{H}_n(x_n) \geq 0$
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## Port-Hamiltonian Theory

$$\underbrace{\begin{pmatrix} \dot{x} \\ w \\ y \end{pmatrix}}_{\text{flows } f} = S \cdot \underbrace{\begin{pmatrix} \nabla \mathcal{H}(x) \\ z(w) \\ u \end{pmatrix}}_{\text{efforts } e}$$

Structuration into

- Constitutive laws :  $\mathcal{H}(x)$  &  $z(w)$
- Power routing paths : Dirac structure  $S$

$$\mathcal{D} = \{(e, f) \in \mathcal{E} \times \mathcal{F} \quad | \quad f = S \cdot e\}$$

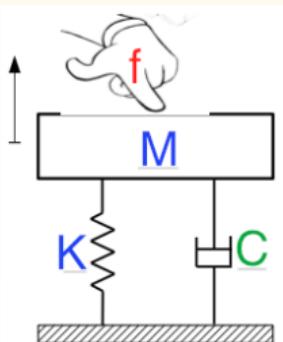
Structurally-guaranteed power balance

$$e^T \cdot f = e^T \cdot S \cdot e = 0 \Leftarrow S \text{ skew-symmetric}$$

Generalizes the state-space representation of linear systems ( $\dot{x} = Ax + Bu$  and  $y = Cx + Du$ )

Applicable to nonlinear multiphysics systems

## Example : damped mechanical oscillator



- Mass : momentum  $\pi$   
kinetic energy  $\mathcal{H}_{kin} = \pi^2/2M$   
with  $\dot{\pi} = \sum f_{ext}$   
 $\partial_\pi \mathcal{H}_{kin} = \pi/M$  (velocity)
- Spring : elongation  $\xi$   
elastic energy  $\mathcal{H}_{el} = K\xi^2/2$  (linear case)  
with  $\dot{\xi} = \Delta v_{ext}$   
 $\partial_\xi \mathcal{H}_{el} = K\xi$  (force)
- Dashpot : Elongation rate  $w = \Delta v_{ext}$  and loss function  $z(w) = Cw = F_{ext \rightarrow C}$
- External operator apply a force  $u = f_{op}$   
and observe the resulting motion  $y = v_{op}$  (at load point)

1. Equating velocities at junction :

$\partial_\pi \mathcal{H} = \dot{\xi}$  (mass-spring),  $\partial_\pi \mathcal{H} = w$  (mass-dashpot) and  $\partial_\pi \mathcal{H} = y$  (mass-ext. op.),

2. Newton's second law :  $\dot{\pi} = \sum f_{ext \rightarrow M} = -\partial_\xi \mathcal{H} - z(w) + u$

Port-Hamiltonian Formulation :

$$\begin{pmatrix} \dot{x} = \begin{pmatrix} \dot{\xi} \\ \dot{\pi} \end{pmatrix} \\ \hline w \\ \hline -y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & -1 & +1 \\ \hline 0 & +1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{S = -S^T} \cdot \begin{pmatrix} \nabla \mathcal{H}(x) = \begin{pmatrix} \partial_\xi \mathcal{H} \\ \partial_\pi \mathcal{H} \end{pmatrix} \\ \hline z(w) \\ \hline u \end{pmatrix}$$

# Discrete-time guaranteed-passive numerical scheme

Standard : finite differences for  $\dot{x}$  and *a posteriori* stability analysis

pHs approach : structure-preserving time-discretization scheme

Chain rule for the time-variation of stored energy :

$$\frac{dE}{dt} = \sum_n \frac{\partial \mathcal{H}}{\partial x_n} \frac{dx_n}{dt} \underset{\nabla^d \mathcal{H}(x[k], \delta x[k])_n}{\approx} \sum_n \underbrace{\frac{\mathcal{H}_n(x_n[k+1]) - \mathcal{H}_n(x_n[k])}{x_n[k+1] - x_n[k]}}_{(\delta x[k]/\delta t)_n} \underbrace{\frac{x_n[k+1] - x_n[k]}{\delta t}}_{(\delta x[k]/\delta t)_n} = \frac{E[k+1] - E[k]}{\delta t}$$

Joint approximation

$$\begin{aligned}\dot{x} &\rightarrow \delta x/\delta t \\ \text{and } \nabla \mathcal{H}(x) &\rightarrow \nabla^d \mathcal{H}(x, \delta x)\end{aligned}$$

$$\rightarrow \underbrace{\begin{pmatrix} \frac{\delta x}{\delta t} \\ w \\ -y \end{pmatrix}}_{f^d} = S \cdot \underbrace{\begin{pmatrix} \nabla^d \mathcal{H}(x, \delta x) \\ z(w) \\ u \end{pmatrix}}_{e^d}$$

Solving  $f^d = S \cdot e^d$  for  $(\delta x, w)$  at each time step  $k$

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Solving  $f^d = S \cdot e^d$  for  $(\delta x, w)$  at each time step  $k$

- Preserves  $S$  and its skew-symmetry  $\Rightarrow \delta E/\delta t + z(w)^T w - u^T y = e^d \cdot f^d = 0$
- Applies directly to nonlinear systems
- Automatic generation of simulation code using the toolbox Python PyPHS  
<https://pypi.python.org/pypi/pyphs>

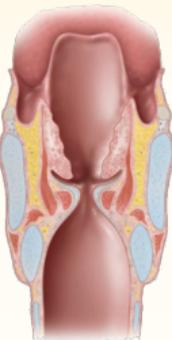
(dev. A. Falaize, IRCAM - LASIE)

## Finite dimension pHs

A minimal model of the vocal apparatus  
Collaboration : Th. Hélie (IRCAM)

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# Physiology, physical principle and components



## Phonation

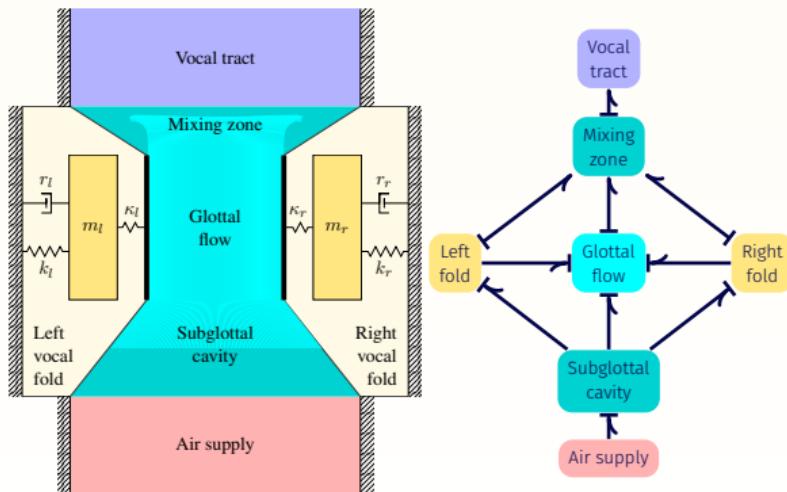
The result of an unstable fluid-structure interaction :

- between the air exhaled from the lungs,
- and the vocal folds that delimitates the glottal constriction.

The vibration of the vocal folds modulate the airflow, which produces acoustic waves within the vocal tract.

## Lumped components :

- Acoustic resonator :  $N$  formants
- Mixing zone : downstream the glottis dissipation of the kinetic energy of jet ( $v > 0$ )
- Vocal folds : mass-spring-damper + elastic cover
- Glottal flow : incompressible potential flow of inviscid gaz, moving walls that pulls and pumps airflow
- Subglottal cavity : division of airflow, pressure forces on the elastic cone
- Ideal Pressure Source



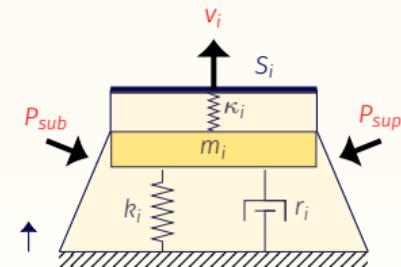
## Vocal folds model

Each fold is modelled as a spring-mass-damping system  
to be coupled to the glottal flow through a cover spring.

- Momentum  $\pi_i$  and elongations  $\xi_i$  and  $\eta_i$  of deep and cover layers

$$\mathcal{H} = \frac{1}{2} \left( \frac{\pi_i^2}{m_i} + k_i \xi_i^2 + \kappa_i \eta_i^2 \right) \Rightarrow \nabla \mathcal{H} = \begin{pmatrix} \pi_i/m_i \\ k_i \xi_i \\ \kappa_i \eta_i \end{pmatrix} = \begin{pmatrix} v_m \\ F_{m \rightarrow k} \\ F_{\kappa \rightarrow m} \end{pmatrix}$$

- Dissipation  $w_i = \delta v = \pi_i/m_i$  and  $z_i(w_i) = F_{m \rightarrow r} = r_i w_i$  (linear dashpot)
- Prescribed velocity  $v_i$  at the glottis port  
Resulting force  $F_i$  of the vocal fold on the glottal flow
- Prescribed upstream pressure  $P_{\text{sub}}$ ,  
resulting flowrate  $Q'_{\text{sub}}$  pulled from the subglottal cavity
- Prescribed downstream pressure ( $P_{\text{sup}}$ ),  
resulting flowrate  $Q'_{\text{sup}}$  pushed into the supraglottal cavity



$$\begin{pmatrix} \dot{\xi}_i \\ \dot{\pi}_i \\ \dot{\eta}_i \\ \hline w_i \\ -Q'_{\text{sub}} \\ Q'_{\text{sup}} \\ F_i \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ -1 & 1 & & & & & \\ & -1 & & & & & \\ \hline & 1 & & & & & \\ & S'_{\text{sub}} & & & & & \\ & S'_{\text{sup}} & & & & & \\ & & & -1 & & & \end{pmatrix} \cdot \begin{pmatrix} \partial \xi_i H \\ \partial \pi_i H \\ \partial \eta_i H \\ \hline z_i(w_i) \\ P_{\text{sub}} \\ P_{\text{sup}} \\ v_i \end{pmatrix}$$

# Glottal flow between two parallel mobile walls

## Objectives

Account for the **transverse velocity**  
and for **power exchanged on the walls**

2D field : Euler equation

$$\nabla \cdot \mathbf{v} = 0 \text{ and } \nabla \times \mathbf{v} = 0 \text{ on } \Omega(t)$$

$$\dot{\mathbf{v}} + \nabla \left( \frac{p}{\rho_0} + \frac{|\mathbf{v}|^2}{2} \right) = 0$$

$$\text{BCs } v_y(y_r) = \dot{y}_r \quad v_y(y_l) = \dot{y}_l$$

$$\text{with } h = y_l - y_r \text{ et } y_m = \frac{y_l + y_r}{2}$$

Kinematics :

$$\mathbf{v} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} + \frac{V_e(t)}{h(t)} \begin{pmatrix} -x \\ y - y_m \end{pmatrix}$$

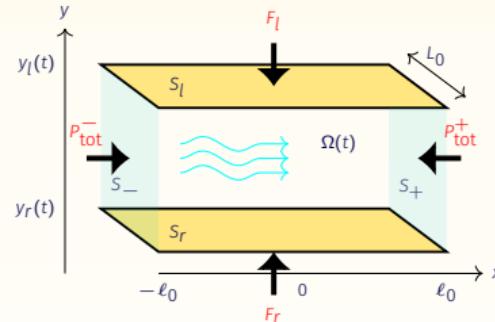
$v_x(t)$  mean axial velocity

$V_y(t) = \dot{y}_m$  mean transverse velocity

$V_e(t) = h$  transverse expansion velocity

$h(t)$  channel height

Related momenta  $\Pi_x$ ,  $\Pi_y$  and  $\Pi_e$



## Dynamics

$$\dot{\Pi}_x(t) = L_0 h_0 (P_{tot}^- - P_{tot}^+)$$

$$\dot{\Pi}_y(t) = \frac{h_0}{h} (F_r - F_l)$$

$$\dot{\Pi}_e(t) = L_0 \ell_0 (P_{tot}^- + P_{tot}^+) + \frac{F_r + F_l}{2} - \partial_h \mathcal{H}$$

$$\dot{h} = \partial_{\Pi_e} \mathcal{H}$$

$$\mathcal{H}(\Pi_x, \Pi_y, \Pi_e, h) = \frac{\Pi_x^2 + \Pi_y^2}{2m(h)} + \frac{\Pi_e^2}{2m_e(h)}$$

# Minimal model for the full vocal apparatus

Independent modelling of each component + interconnection

$$\mathcal{H} = \mathcal{H}_r(\pi_r, \xi_r, \eta_r) + \mathcal{H}_l(\pi_l, \xi_l, \eta_l) + \mathcal{H}_{\text{glottis}}(\Pi_x, \Pi_y, \Pi_e, h) + \mathcal{H}_{\text{ac}}(P_{\text{ac}}, Y_{\text{ac}})$$

Right fold	$\begin{pmatrix} \dot{\pi}_r \\ \dot{\xi}_r \\ \dot{\eta}_r \end{pmatrix}$	$\begin{pmatrix} -r_r & -1 & 1 \\ 1 & & \\ -1 & & \end{pmatrix}$	$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$	$\begin{pmatrix} -S_{\text{sup}}^d 1_N^T \\ & \end{pmatrix}$	$\begin{pmatrix} -S_{\text{sub}}^d \\ & \end{pmatrix}$	$\begin{pmatrix} \partial_{\pi_r} \mathcal{H} \\ \partial_{\xi_r} \mathcal{H} \\ \partial_{\eta_r} \mathcal{H} \end{pmatrix}$
Left fold	$\begin{pmatrix} \dot{\pi}_l \\ \dot{\xi}_l \\ \dot{\eta}_l \end{pmatrix}$	$\begin{pmatrix} -r_l & -1 & 1 \\ 1 & & \\ -1 & & \end{pmatrix}$	$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$	$\begin{pmatrix} -S_{\text{sup}}^g 1_N^T \\ & \end{pmatrix}$	$\begin{pmatrix} -S_{\text{sub}}^g \\ & \end{pmatrix}$	$\begin{pmatrix} \partial_{\pi_l} \mathcal{H} \\ \partial_{\xi_l} \mathcal{H} \\ \partial_{\eta_l} \mathcal{H} \end{pmatrix}$
Glottis	$\begin{pmatrix} \dot{\Pi}_x \\ \dot{\Pi}_y \\ \dot{\Pi}_e \\ \dot{h} \end{pmatrix}$	$\begin{pmatrix} h_0 \\ \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$	$\begin{pmatrix} -L_0 h_0 1_N^T \\ L_0 \ell_0 1_N^T \\ 1 \end{pmatrix}$	$\begin{pmatrix} -L_0 h_0 \\ L_0 \ell_0 \\ L_0 \ell_0 \end{pmatrix}$	$\begin{pmatrix} \partial_{\Pi_x} \mathcal{H} \\ \partial_{\Pi_y} \mathcal{H} \\ \partial_{\Pi_e} \mathcal{H} \\ \partial_h \mathcal{H} \end{pmatrix}$
Acous.	$\begin{pmatrix} \dot{P}_{\text{ac}} \\ \dot{Y}_{\text{ac}} \end{pmatrix}$	$S_{\text{sup}}^d 1_N$	$S_{\text{sup}}^g 1_N$	$L_0 h_0 1_N$	$\begin{pmatrix} -R_{\text{ac}} & -\mathbb{I}_N \\ \mathbb{I}_N & \end{pmatrix}$	$\begin{pmatrix} \nabla_{P_{\text{ac}}} \mathcal{H} \\ \nabla_{Y_{\text{ac}}} \mathcal{H} \end{pmatrix}$
Mixing Output	$\begin{pmatrix} W_{\text{turb}} \\ -Q_{\text{sub}} \end{pmatrix}$	$S_{\text{sub}}^d$	$S_{\text{sub}}^g$	$\begin{pmatrix} L_0 h_0 & -L_0 \ell_0 \\ -L_0 h_0 & -L_0 \ell_0 \end{pmatrix}$	$\begin{pmatrix} & \\ & \end{pmatrix}$	$\begin{pmatrix} Z_{\text{turb}} \\ P_{\text{sub}} \end{pmatrix}$

- Sparse interconnection matrix  $S$  (routing between neighbours only)
- Elimination of linear losses → Negative symmetric part ( $a_r, a_l, R_{\text{ac}}$ )
- Nonlinearities hidden :
  1. in the Hamiltonian of the glottal flow (non quadratic  $\mathcal{H}_{\text{glottis}}$ );
  2. in the lossy mixing region ( $Z_{\text{turb}}$ ) downstream the glottis;
  3. possibly in nonlinear constitutive law of soft tissues
- Non full rank → easy detection of invariants (here  $h + \eta_r + \eta_l + \xi_r + \xi_l = \text{cst}$ )

# First results of time-domain simulations

Parameters values from literature :

Folds :  $m_i = 0.2 \text{ g}$ ,  $r_i = 0.05 \text{ N s m}^{-1}$ ,  $\kappa_i = 3k_i$ ,  
 $S_{\text{sup}}^i = 1.1 \text{ mm}^2$ ,  $S_{\text{sub}}^i : 1.1 \text{ cm}^2$

Glottis :  $L_0 = 11 \text{ mm}$ ,  $2\ell_0 = 4 \text{ mm}$

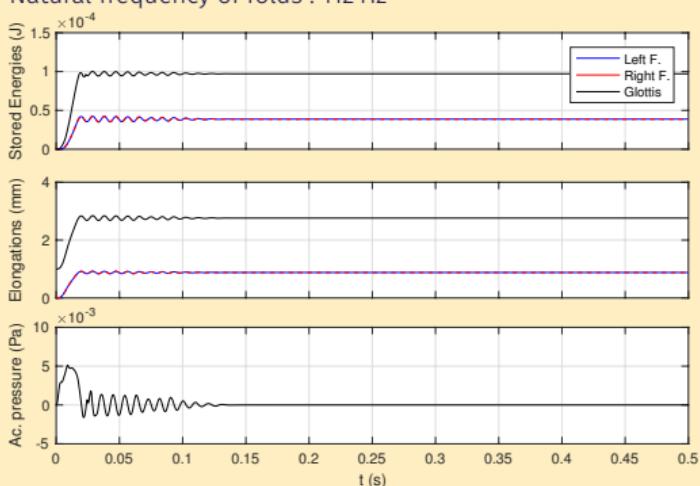
Vocal tract : R1 /a/       $f_0 = 640 \text{ Hz}$ ,  $q_0 = 2.5$ ,  $Z_0 = 1 \text{ M}\Omega$

Input  $P_{\text{sub}}$  :  $0 \rightarrow 800 \text{ Pa}$  within 20 ms

## Open glottis ( $h_r = 1 \text{ mm}$ )

$$k_r = k_l = 100 \text{ N m}^{-1}$$

Natural frequency of folds : 112 Hz



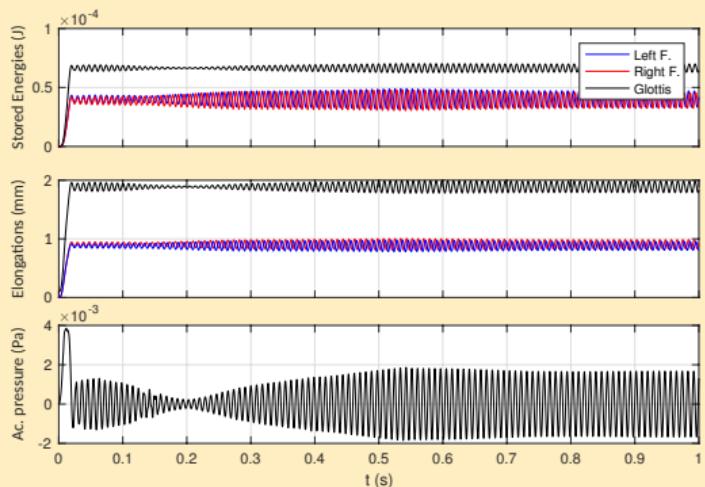
No oscillation onset

## Adduction (nearly closed, $h_r = 0.1 \text{ mm}$ )

Slightly detuned vocal folds

$$k_l = 100 \text{ N m}^{-1} \text{ (natural freq. } 112 \text{ Hz)}$$

$$k_r = 97 \text{ N m}^{-1} \text{ (natural freq. } 110 \text{ Hz)}$$



Periodic oscillations :

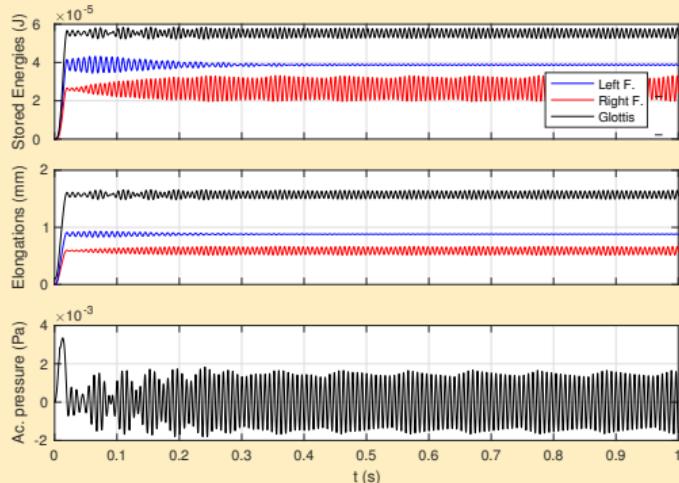
- Oscillation stabilized after some transient
- Synchronized folds vibrations even without contact between folds

# First results of time-domain simulations

Strong asymmetry (still nearly closed,  $h_r = 0.1 \text{ mm}$ )

$k_l = 100 \text{ N m}^{-1}$  (natural freq. 112 Hz)

$k_r = 149 \text{ N m}^{-1}$  (natural freq. 137 Hz)



Quasi-periodic oscillations :

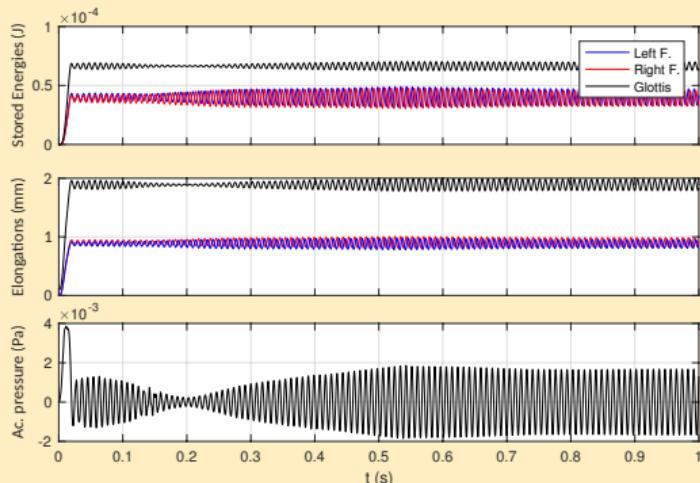
- starting on the left (lax) vocal fold,
- transferred to the right (stiff) fold before reaching the steady state regime.

Adduction (nearly closed,  $h_r = 0.1 \text{ mm}$ )

Slightly detuned vocal folds

$k_l = 100 \text{ N m}^{-1}$  (natural freq. 112 Hz)

$k_r = 97 \text{ N m}^{-1}$  (natural freq. 110 Hz)



Periodic oscillations :

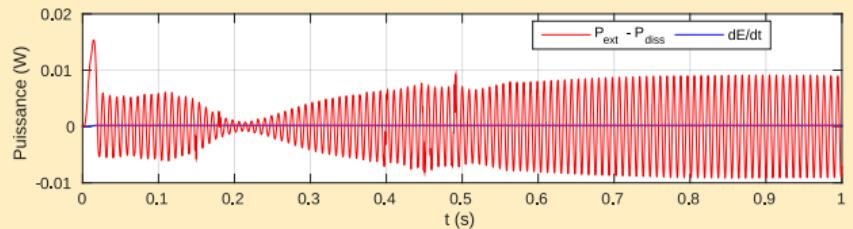
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- Synchronized folds vibrations even without contact between folds

Previous results were computed using standard integrator (Matlab ode15s)

## Power balance

Power is arbitrarily generated or consumed by the numerical scheme

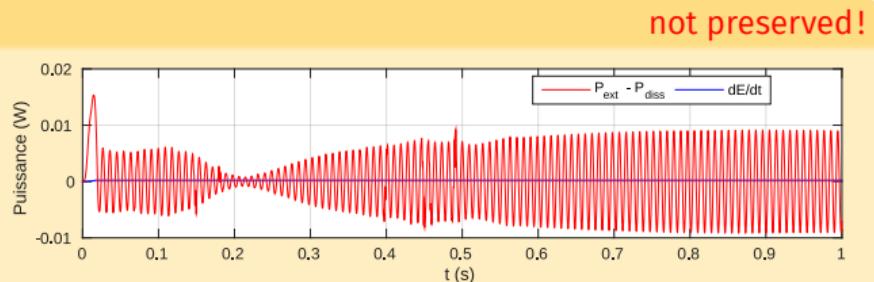
not preserved!



Previous results were computed using standard integrator (Matlab ode15s)

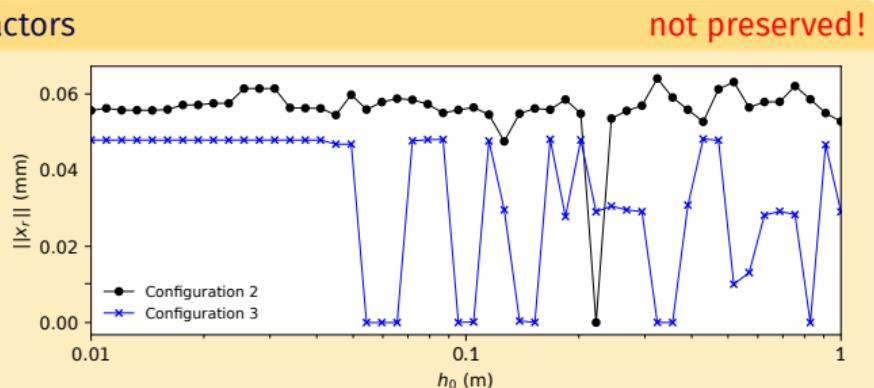
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### Invariance of the results w.r.t scaling factors

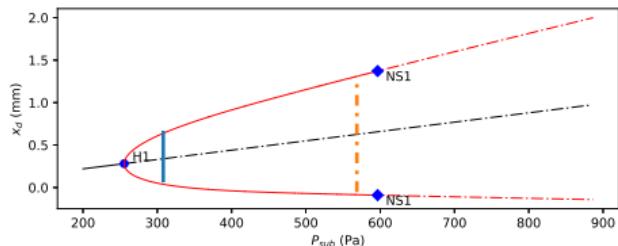
$h_0$  : arbitrary scaling factor used in the definition of the glottal momenta



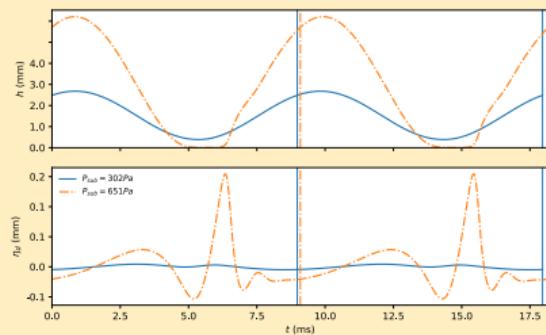
PyPHS numerical simulations not affected (also proven by formal analysis of the numerical scheme)  
 → Results obtained with ode15s/23/23tb/... are **not reliable**!

# Bifurcation analysis

Control parameter : subglottal pressure  $P_{sub}$   
Static sol. (black) / Limit cycles (red, min/max)



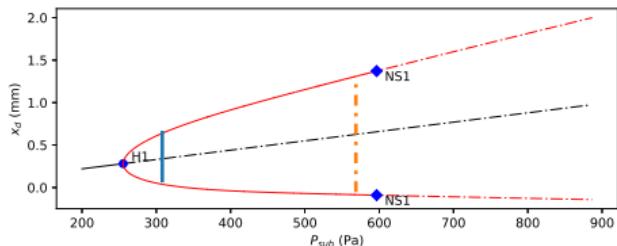
## Hopf bifurcation & limit cycles



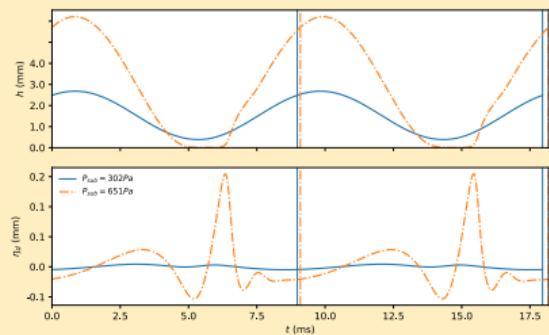
Onset pressure close to the one obtained from time-domain simulation using the numerical scheme from PyPHS.  
Nearly sinusoidal oscillation near threshold

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Static sol. (black) / Limit cycles (red, min/max)



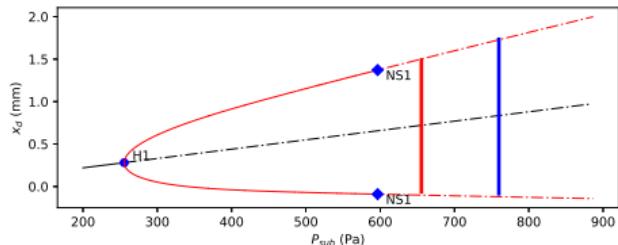
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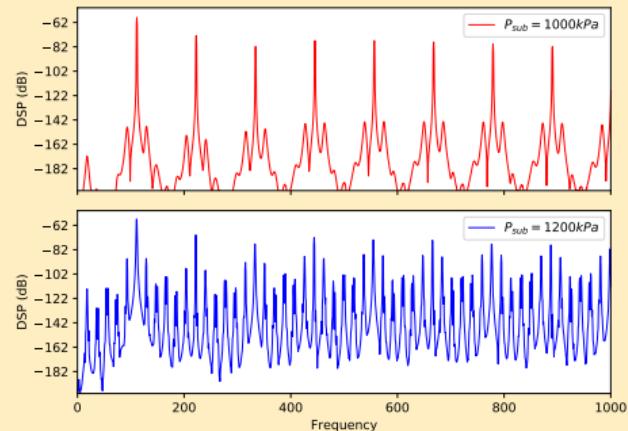
For finite displacements, (almost) contact between folds.  
Evaluation of this contact force from the compression of cover  
→ Possibly correlated to phonotraumatism

# Bifurcation analysis

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Static sol. (black) / Limit cycles (red, min/max)



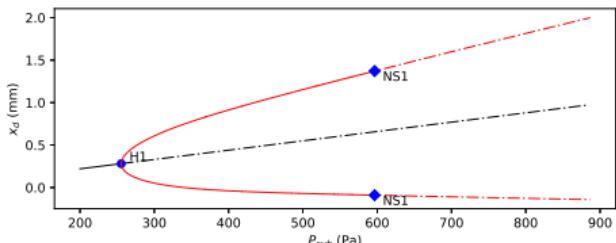
## Neimark-Sacker bifurcation



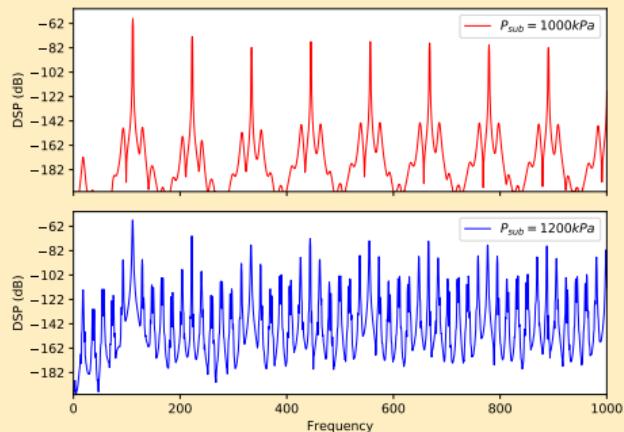
Raucous sounds (QP) for very high pressures :  
Qualitatively agree with shouted voice measurements

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## Neimark-Sacker bifurcation



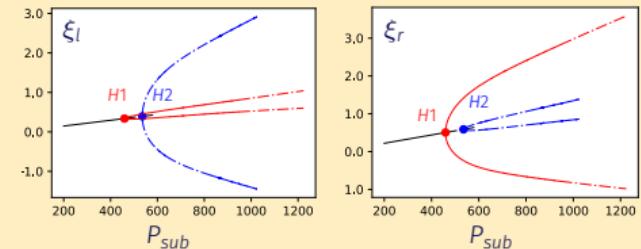
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## Asymmetric vocal folds

Bifurcation diagram for  $\xi_l$  and  $\xi_r$

$$k_l = 150 \text{ N/m}$$

$$k_r = 100 \text{ N/m}$$

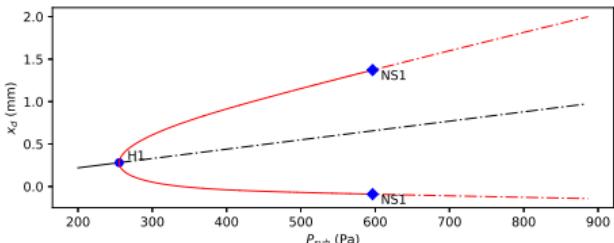


Two limit cycles branches

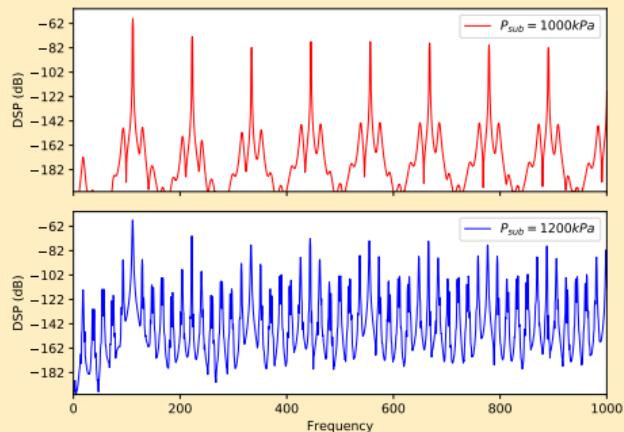
Interleaved stability regions → dysphonia ?

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## Neimark-Sacker bifurcation



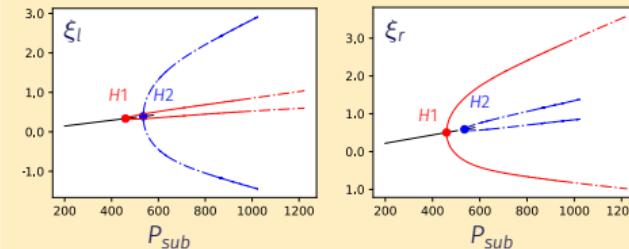
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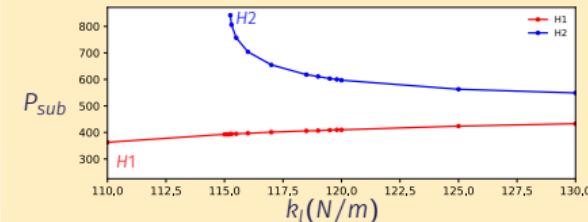


Two limit cycles branches

Interleaved stability regions → dysphonia ?

Continuation of Hopf bifurcations

2<sup>nd</sup> Control parameter : stiffness  $k_l$



2<sup>nd</sup> branch appears above an asymmetry threshold

Infinite dimension pHs

Application to the vocal tract

Coll. : Th. Hélie (IRCAM), V. Wetzel (PhD)

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## Extension to distributed systems with boundary control

- Discrete internal variables  $x(t) \rightarrow$  Fields  $x(M, t)$  (i.e.,  $k$ -forms using the terminology of differential geometry)
- Hamiltonian  $\mathcal{H} = \int_{\Omega} H(x(M, t)) d\Omega$  with  $H$  the energy density;
- Gradient  $\nabla_x \mathcal{H} \rightarrow$  variational derivative  $\delta_x H$ ;
- Evolution equation (ODE)  $\rightarrow$  Local equations (PDE) :

Strong form :  $\partial_t x = \mathcal{J} \delta_x H$

$$\mathcal{J} = \sum_{d=0}^D A_d \frac{\partial^d}{\partial z^d} \text{ with } A_d = (-1)^{d+1} A_d^T$$

where  $\mathcal{J}$  holds the differential operations (e.g., diffusion);

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- What about boundary conditions?  $\rightarrow$  Weak form

$$\begin{aligned} \forall \text{ test field } \bar{\mathbf{x}}, \int_{\Omega} \bar{\mathbf{x}} \cdot \partial_t \mathbf{x} &= \int_{\Omega} \bar{\mathbf{x}} \cdot \mathcal{J} \delta_{\mathbf{x}} H = \int_{\Omega} \bar{\mathbf{x}} \cdot \bar{\mathcal{J}} \delta_{\mathbf{x}} H + \int_{\partial\Omega} \bar{\mathbf{x}} \cdot \bar{\mathcal{B}} \mathbf{u} \\ \text{or, formally, } \partial_t \mathbf{x} &= \bar{\mathcal{J}} \delta_{\mathbf{x}} H + \bar{\mathcal{B}} \mathbf{u} \end{aligned}$$

Stockes' theorem is applied (partially) in order to introduce boundary control (chosen inputs).

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- 3-steps Space-time Discretization

1. Discretization of the Stockes-Dirac structure ( $\mathcal{J}$ )
2. Discretization of the variables (fields)  $\mathbf{x}$  and of the Hamiltonien  $\mathcal{H}$
3. Time discretization of the finite dim. pHS resulting from the previous steps

# Application to 1D wave equations

Linear vibrating string ( $z \in (0, L)$ )

$$\rho(z) \partial_{tt} u_y(t, z) = T_0 \partial_{zz} u_y$$

$u_y(t, z)$  transverse displacement

$\rho(z)$  linear mass density

$T_0$  tension of the string (hyp. small vibrations)

Inertia  $\rightarrow$  momentum  $\pi = \rho \partial_t u_y$

Elasticity  $\rightarrow$  strain  $\xi = \partial_z u_y$

Hamiltonian :

$$\mathcal{H}(\pi, \xi) = \int_0^L \frac{1}{2} \frac{\pi^2}{\rho} + \frac{1}{2} T_0 \xi^2 dz$$

i.e.

$\delta_\pi H = \pi/\rho = \partial_t u_y$  transverse velocity

$\delta_\xi H = T_0 \xi = T_0 \partial_z u_y$  transverse force ( $z^+ \rightarrow z^-$ )

Distributed pHS model

$$x = \begin{pmatrix} \pi \\ \xi \end{pmatrix} \quad \text{et} \quad \frac{\partial x}{\partial t} = \begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} \cdot \frac{\delta H}{\delta x}$$

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Some classical extensions :

• Variable tension  $T(\xi) = T_0 + ES\xi^2$

$$\rho(z) \partial_{tt} u_y(t, z) = \partial_z \left( T_0 \partial_z u_y + ES (\partial_z u_y)^3 \right)$$

$\mathcal{H}(\pi, \xi)$  is not quadratic anymore

No change on  $\mathcal{J}$

• Stiff string (e.g., Issanchou et al, JASA 2018)

$$\rho(z) \partial_{tt} u_y(t, z) = T_0 \partial_{zz} u_y - EI \partial_z^{(4)} u_y$$

Additional variable :  $\kappa = -\partial_{zz} u_y$  (curvature)

$$\mathcal{H}(\pi, \xi, \kappa) = \int_0^L \frac{1}{2} \frac{\pi^2}{\rho} + \frac{1}{2} T_0 \xi^2 dz + \frac{1}{2} EI \kappa^2 dz$$

$$\frac{\partial x}{\partial t} = \underbrace{\begin{pmatrix} \partial_z & \partial_{zz} \\ -\partial_{zz} & \end{pmatrix}}_{\mathcal{J} \text{ order 2}} \begin{pmatrix} \delta_\pi \mathcal{H} \\ \delta_\xi \mathcal{H} \\ \delta_\kappa \mathcal{H} \end{pmatrix}$$

• Longitudinal motion of the string

$\rightarrow$  Kirchoff-Carrier model

New variables  $\pi_z = \rho \partial_t u_z$  and  $\xi_z = \partial_z u_z$

Debut & Antunes : pitch glide effect of the portuguese guitar

## Some examples in structural mechanics

Euler-Bernoulli Beam  $\rho \partial_{tt} u_y = EI \partial_x^{(4)} u_y$

$$x = \begin{pmatrix} \pi \\ \kappa \end{pmatrix} = \begin{pmatrix} \rho \partial_t u_y \\ -\partial_{zz} u_y \end{pmatrix}, \mathcal{H}(x) = \int_0^L \frac{1}{2} \left( \frac{\pi^2}{\rho} + EI\kappa^2 \right) dz$$

$$\partial_t x = \begin{pmatrix} \partial_{zz} & \partial_{zz} \end{pmatrix} \delta_x H \text{ with } \delta_x H = \begin{pmatrix} \pi/\rho \\ EI\kappa \end{pmatrix} \begin{array}{l} \text{velocity} \\ \text{bending m.} \end{array}$$

Timoshenko beam

$$x = \begin{pmatrix} \pi_y \\ \pi_\phi \\ \partial_x \phi \\ \gamma \end{pmatrix} \begin{array}{l} \text{momentum} \\ \text{ang. mom.} \\ \text{shear strain} \end{array}$$

$$\delta_x H = \begin{pmatrix} \partial_t u_y \\ \partial_t \phi \\ M \\ T \end{pmatrix} \begin{array}{l} \text{vel.} \\ \text{ang. vel.} \\ \text{bend. moment} \\ \text{shear force} \end{array}$$

$$\partial_t x = \begin{pmatrix} & \partial_z & \partial_z \\ \partial_z & \partial_z & -1 \end{pmatrix} \delta_x H$$

Quite differently structured

Kirchhoff plate

2D extension of the Bernoulli beam

$x = (\pi, \mathbb{K})$  where  $\mathbb{K}$  is the curvature tensor

$$\mathcal{H}(x) = \iint \frac{1}{2} \frac{\pi^2}{\rho} + \frac{1}{2} \mathbb{K} : \mathbb{D}\mathbb{K}$$

then

$$\partial_t x = \begin{pmatrix} 0 & -\text{div} \circ \text{Div} \\ \text{Grad} \circ \text{grad} & 0 \end{pmatrix} \delta_x H$$

where  $\text{div} \circ \text{Div}$  and its adjoint  $\text{Grad} \circ \text{grad}$  degenerate in  $\partial_{zz}$  in 1D.

## Time-space discretization

1. Discretization of the Stockes-Dirac structure ( $\mathcal{J}$ )
2. Discretization of the variables (fields)  $x$  and of the Hamiltonien  $\mathcal{H}$
3. Time discretization of the finite dim. pHs resulting from the previous steps

Structure  $f = \mathcal{J}e$

$$\text{Power flowing : } \mathcal{P} = \int_{\Omega} e \cdot f \, d\Omega = \sum_i \int_{\Omega} e^i f^i \, d\Omega$$

Focusing on  $f = A_1 \partial_z e$  (other cases follow from this)

Satisfied locally for  $f \in \mathbb{P}^d$  and  $e \in \mathbb{P}^{d+1}$

$$f^i(t, z) = \sum_{n=1}^N F_n^i(t) \Psi_n(z) \text{ and } e^i(t, z) = \sum_{m=1}^{N+1} E_m^i(t) \Phi_m(z)$$

with  $\forall m, \partial_z \Phi_m \in \text{span}(\Psi_1, \dots, \Psi_N)$

→ No power-pairing :  $\mathcal{P} \neq F \cdot E!$

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$(N+1)l$  coeffs.  $E_m^i = (NI \text{ dofs } \bar{E}_n^i) + (l \text{ inputs on boundary})$

Bassi et al, LNCIS 366, p. 61-73, 2007

Kind of double discretization of efforts

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Kind of double discretization of efforts

Application to the vibrating string  $e = (v, F)$   
 $f$  piecewise constant (=  $F_n^i$  on element  $T_n$ )  
 $e$  piecewise linear ( $E_m^i$  are the nodal values)  
 $\rightarrow \bar{E}_n^i = (E_n^i + E_{n+1}^i) / 2$

Selecting two inputs, for example :

velocity at $z = 0$ :	$v(0) = e^1(0) = E_1^1$
force at $z = L$ :	$F(L) = e^2(L) = E_{N+1}^2$

Dofs :  $\bar{E} = (\bar{E}_1^1 \dots \bar{E}_N^1, \bar{E}_1^2 \dots \bar{E}_N^2)$

Inputs :  $u = (v(0), F(L))$

Outputs :  $y = (F(0), -v(L))$  (generator conv.)

$$\begin{pmatrix} (F_n^1)_{n=1:N} \\ (F_n^2)_{n=1:N} \\ y^1 = F_{z=0} \\ y^2 = -v_{z=L} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{U} & 0 & \mathbb{C}_f \\ -\mathbb{U}^T & 0 & \mathbb{C}_v & 0 \\ 0 & -\mathbb{C}_v^T & 0 & -1 \\ -\mathbb{C}_f^T & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} (\bar{E}_n^1)_{n=1:N} \\ (\bar{E}_n^2)_{n=1:N} \\ u^1 = v_{z=0} \\ u^2 = F_{z=L} \end{pmatrix}$$

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Structure  $f = \mathcal{J}e$

$$\begin{pmatrix} (F_n^1)_{n=1:N} \\ \vdots \\ (F_n^l)_{n=1:N} \\ (y^i)_{i=1:l} \end{pmatrix} = S \cdot \begin{pmatrix} (\bar{E}_n^1)_{n=1:N} \\ \vdots \\ (\bar{E}_n^l)_{n=1:N} \\ (u^i)_{i=1:l} \end{pmatrix}$$

Hamiltonien  $\mathcal{H} = \int_{\Omega} H(x(M, t)) d\Omega$

Discretization of  $x$  similar to  $f \rightarrow x^d = (x_{n=1:N}^i)_{i=1:l}$

Discrete Hamiltonian  $\mathcal{H}^d(x^d) = \mathcal{H} \left( \sum_{n=1}^N x_n^i \psi_n(z) \right)$

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Discrete pHs model

With  $f^i = \dot{\mathbf{x}}_i$  and  $\bar{e}^i = \partial_{\mathbf{x}_i} \mathcal{H}^d$

$$\begin{pmatrix} (\dot{\mathbf{x}}_n^1)_{n=1:N} \\ \vdots \\ (\dot{\mathbf{x}}_n^l)_{n=1:N} \\ (y^i)_{i=1:l} \end{pmatrix} = S \cdot \begin{pmatrix} (\partial_{\mathbf{x}_n^1} \mathcal{H})_{n=1:N} \\ \vdots \\ (\partial_{\mathbf{x}_n^l} \mathcal{H})_{n=1:N} \\ (u^i)_{i=1:l} \end{pmatrix}$$

+ Finite dim. pHs time-discretization scheme

For the vibrating string :

$$\begin{pmatrix} (\dot{\pi}^i)_{i=1:N} \\ (\dot{\xi}^i)_{i=1:N} \\ F(0) \\ -v(L) \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{U} & 0 & \mathbb{C}_f \\ -\mathbb{U}^T & 0 & \mathbb{C}_v & 0 \\ 0 & -\mathbb{C}_v^T & 0 & -1 \\ -\mathbb{C}_f^T & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \nabla_{\pi_i} \mathcal{H}^d \\ \nabla_{\xi_i} \mathcal{H}^d \\ v(0) \\ F(L) \end{pmatrix}$$

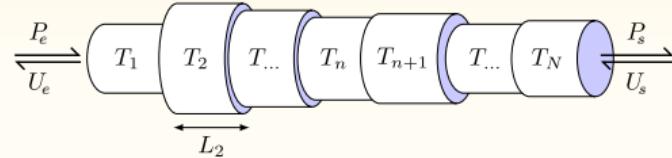
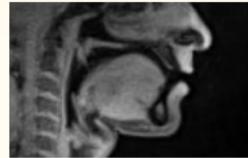
Kind of mixed finite elements formulation

Hamiltonien  $\mathcal{H} = \int_{\Omega} \mathcal{H}(\mathbf{x}(M, t)) d\Omega$

Discretization of  $\mathbf{x}$  similar to  $f \rightarrow \mathbf{x}^d = (\mathbf{x}_{n=1:N}^i)_{i=1:l}$

Discrete Hamiltonian  $\mathcal{H}^d(\mathbf{x}^d) = \mathcal{H} \left( \sum_{n=1}^N \mathbf{x}_n^i \Psi_n(z) \right)$

# Ongoing work : modelling the vocal tract with articulators



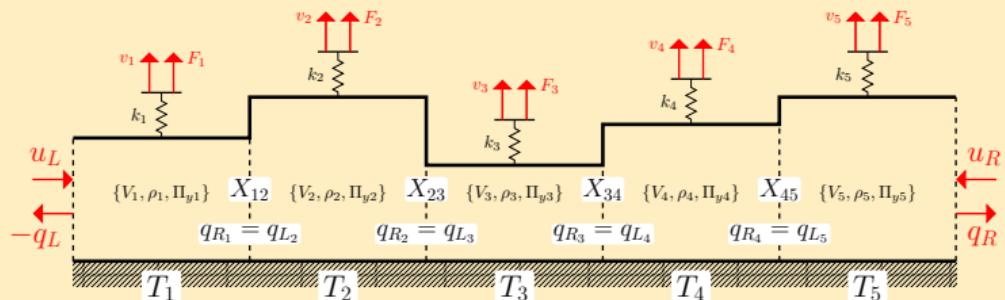
Classical approach : rigid wall acoustics with time-varying section (i.e., no effective power exchange)

Within the PhD of Victor Wetzel (coadvisor Th. Hélie, IRCAM)

Accounting for :

- elasticity of soft tissues
- motion of the articulators (mouth, lips, jaw, velum)

Distributed pHs with boundary control



Finite amplitude motion of the articulators hardly compatible with the hypothesis of linear acoustics.

→ Back to fluid and thermodynamics

Communication at ICSV2019, Montréal ([Wetzel](#), Hélie et Silva).

## Conclusion

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# A growing community focusing on pHs

- 1990 First papers on «*Port-controlled Hamiltonian systems*» (van der Schaft)  
Exploit structure of systems for control purposes (stabilization, optimal control, etc...)  
Application to conservative, dissipative, discrete, constrained, commuted, nonlinear, nonconvex, ...systems
- 2000 Fundamental paper on distributed pHs  
«*Hamiltonian formulation of distributed-parameter systems with boundary energy flow*»,  
A. van der Schaft et B. Maschke, J. Geom. Phys 42, 166–194, 2002
- 2005 First papers on the space discretization of distributed pHs
- 2011-16 ANR HAMECMOPSY  
(*Hamiltonian Methods for the Control of Multidomain Distributed Parameter Systems*),  
LAGEP, FEMTO-st, ISAE, IECN. Acoustics, structural mechanics, thermics, chemistry.
- 2011-16 ERC MODSIMCONMP  
(*Modeling, Simulation and Control of Multi-Physics Systems*, 2M€)
- 2017-20 ANR-DFG INFIDHEM  
(*Interconnected infinite-dimensional systems for heterogeneous media*)

## Concluding remarks & Perspectives

### A physics-based Nonlinear "State-Space" Representation

- modular : model and simulate the components (almost) independently
- essential focus on energy and power balance
- with benefits on :
  - the time-domain simulation
  - the design of controller to achieve target behaviour (energy shaping)

### Increase usability of pHs tools :

- Higher order for space discretization
- PyPHS limited to finite dimension pHs by now.
  - Allow specification of distributed systems + simulation

### Bridge the gap between control-oriented modelling and NL dynamics

- exploit structure for modal and bifurcation analysis
- introduce structure in NL system identification
- preserve (some) structure in model order reduction