

On a class of quasilinear Barenblatt equations

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This paper is dedicated to Prof. Monique Madaune Tort.

Abstract

We first investigate the following quasilinear parabolic equation of Barenblatt type,

$$\begin{cases} f(\cdot, \partial_t u) - \Delta_p u - \epsilon \Delta(\partial_t u) = g & \text{in } Q = \Omega \times]0, T[\\ u = 0 & \text{on } \Gamma = \partial\Omega, \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases} \quad (\text{P}_t)$$

where Ω is a bounded domain with Lipschitz boundary, denoted by Γ in \mathbb{R}^d with $d \geq 1$, $\frac{2d}{d+2} < p < \infty$, $\epsilon \geq 0$, $0 < T < +\infty$, $u_0 \in W_0^{1,p}(\Omega)$ and f is a Carathéodory function which satisfies suitable growth conditions and $g \in L^2(Q)$. We prove the existence of a weak solution (see definition 1.1) and give some related regularity results. Next, we analyse further the case $p = 2$ with $\epsilon = 0$. Precisely, we are concerned by the study of a Barenblatt problem involving a stochastic perturbation:

$$\begin{cases} f\left(\partial_t\left(u - \int_0^t h dw\right)\right) - \Delta u = 0 & \text{in } Q \times \Theta \\ u = 0 & \text{on } \partial\Omega, \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases} \quad (\text{S}_t)$$

where $\int_0^t h dw$ denotes the Itô integral of h and $f : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing bi-Lipschitz continuous function. (Θ, \mathcal{F}, P) is the probability space. Under these conditions, we prove the existence and the uniqueness of the weak solution (see definition 1.4).

Keywords: Quasilinear parabolic equation, Barenblatt equation, time-semi-discretization, stochastic perturbation, pseudo-parabolic equation, p -Laplace operator

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Introduction

In this paper we first investigate the following quasilinear parabolic problem of the Barenblatt type :

$$\begin{cases} f(\cdot, \partial_t u) - \Delta_p u - \epsilon \Delta(\partial_t u) = g & \text{in } Q =]0, T[\times \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases} \quad (\text{P}_t)$$

where Ω is a bounded domain with Lipschitz boundary, denoted by Γ in \mathbb{R}^d with $d \geq 1$, $\frac{2d}{d+2} < p < \infty$, $\epsilon \geq 0$, $0 < T < +\infty$, $u_0 \in W_0^{1,p}(\Omega)$ and f is a Carathéodory function which satisfies $f(x, 0) = 0$ and suitable growth assumptions and $g \in L^2(Q)$. We stress that when $\epsilon = 0$, f satisfies additionally monotonicity assumptions. We look for *weak* solutions (solutions, for short) of Problem (P_t) , that is, for functions satisfying Definition 1.1 and we discuss the following issues: uniqueness and regularity of solutions. Next, we focus on the nondegenerate case $p = 2$ and $\epsilon = 0$. Precisely we are concerned with the following Barenblatt equation involving a stochastic perturbation:

$$\begin{cases} f\left(\partial_t\left(u - \int_0^t h dw\right)\right) - \Delta u = 0 & \text{in } Q \times \Theta \\ u = 0 & \text{on } \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases} \quad (\text{S}_t)$$

where $\int_0^t h dw$ denotes the Itô integral of h and $f : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing Lipschitz function. (Θ, \mathcal{F}, P) is the probability space. We are able to discuss the existence and the uniqueness of weak solutions even in the case where h depends on u . The notion of weak solutions to (S_t) is given in Definition 1.4.

These classes of Barenblatt equations were originally considered by G.I. BARENBLATT in [8]. In S. KAMIN-L.A. PELETIER-J.L. VÁZQUEZ [18], the authors establish the existence of self-similar solutions of a Barenblatt equation arising in porous media models. The existence of self-similar solutions for a class of quasilinear degenerate Barenblatt equations related to porous media problems were further investigated in J. HULSHOF-J.L. VÁZQUEZ [16] and in N. IGBIDA [17]. Barenblatt type problems appear also in a wide variety of situations in Physics, Biology and Engineering. In particular, in P. COLLI-F. LUTEROTTI-G. SCHIMPERNA-U. STEFANELLI [9] a pseudo-parabolic Barenblatt equation motivated by an irreversible phase change model is studied and in M. PTASHNYK [21] the analysis of similar kind of equations is used for reaction-diffusion with absorption problems in Biochemistry. In the context of constrained stratigraphic problems in Geology, the study of Barenblatt equations were recently revisited by different authors. In this regard, we can quote the contributions S.N. ANTONTSEV-G. GAGNEUX-R. LUCE-G. VALLET [4], [5], [6], [7], G. VALLET [25] and for related problems with stochastic coefficients ADIMURTHI-SEAM NGONN-G. VALLET [1]. Finally, in G. DIAZ-J.I. DIAZ [12] and in K.S. HA [15] the existence of solutions to a class of homogeneous quasilinear Barenblatt equations is established by means of monotone methods for m -accretive operators. In the present work, we study

further the class of quasilinear and pseudo-parabolic Barenblatt equations involving a p -Laplace operator. We stress that we obtain existence results for a larger class of functions f in respect to the previous works. In particular, we do not require f to be nondecreasing.

This paper is organised as follows. The next section (Section 1) contains some classical notations used throughout the paper and the statements of our main results on the solvability of problems (P_t) and (S_t) , Theorems 1.2 and 1.5. The proof of Theorem 1.2 is established in Section 2 whereas in Section 3, Theorem 1.5 is proved.

1 Main Results

First, we introduce some notation which will be used throughout the paper. We denote by $X = W_0^{1, \max(2, p)}(\Omega)$ and

$$a_p : W_0^{1, p}(\Omega) \times W_0^{1, p}(\Omega) \rightarrow \mathbb{R}, \quad (u, v) \rightarrow \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

the principal eigenvalue of $-\Delta_p$ in Ω is denoted by $\lambda_1(p, \Omega)$.

For any $N > 0$, we set $\Delta t = \frac{T}{N}$ and $t_k = k\Delta t$. Consequently for any sequence $(v^k)_{k \in \mathbb{N}} \subset W_0^{1, p}(\Omega)$, we define:

$$v^{\Delta t} = \sum_{k=0}^{N-1} v^{k+1} \mathbf{1}_{[t_k, t_{k+1}[} \quad \text{and} \quad \tilde{v}^{\Delta t} = \sum_{k=0}^{N-1} \left[\frac{v^{k+1} - v^k}{\Delta t} (t - t_k) + v^k \right] \mathbf{1}_{[t_k, t_{k+1}[}.$$

Let E, F be separable Banach spaces with $E \hookrightarrow F$ and $1 < p < +\infty$. We denote $L^p(0, T; E)$ the space of E -valued measurable functions u such that $t \rightarrow \|u(t)\|_E$ belongs to $L^p(0, T)$. Furthermore,

$$W^{1, p_1, p_2}(0, T; E; F) \stackrel{\text{def}}{=} \{u \in L^{p_1}(0, T; E) \mid \partial_t u \text{ (in the sense of distributions)} \in L^{p_2}(0, T; F)\} \\ \text{and } H^1(0, T; E; F) \stackrel{\text{def}}{=} W^{1, 2, 2}(0, T; E; F).$$

Regarding Problem (P_t) , we now describe the assumptions required about f :

- (f1) $f : (x, t) \in \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(\cdot, 0) = 0$;
- (f2) $\exists c_1 \in L^2(\Omega)$, $c_2 \geq 0$ such that $|f(x, t)| \leq c_1(x) + c_2|t|$, $x \in \Omega$ a.e. and for any $t \in \mathbb{R}$;
- (f3) $A_\epsilon : L^2(\Omega) \rightarrow L^2(\Omega)$, $u \mapsto f(\cdot, u) - \epsilon \Delta u$ is monotone*.

If $\epsilon > 0$ we suppose in addition that

- (f4) $\exists C_\epsilon < \epsilon \lambda_1(2, \Omega)$ and $C_1 \in L^2(\Omega)$ such that $f(x, t) \text{sign}(t) \geq -C_\epsilon |t| - C_1(x)$, $\forall t \in \mathbb{R}$ and $x \in \Omega$ a.e.

If $\epsilon = 0$, we assume the following condition:

- (f5) $\exists C > 0$ and $C_1 \in L^2(\Omega)$ such that $f(x, t) \text{sign}(t) \geq C|t| - C_1(x)$, $\forall t \in \mathbb{R}$ and $x \in \Omega$ a.e.[†]

*Remark that $t \mapsto f(\cdot, t)$ has not to be a nondecreasing function when $\epsilon > 0$.

†Note that (f5) corresponds to (f4) with $C_\epsilon = C_0 = -C < 0$.

First, we give the notion of weak solution of (P_t) :

Definition 1.1. If $\epsilon > 0$, a weak solution of (P_t) is any $u \in W^{1,p,2}(0, T; W_0^{1,p}(\Omega), H_0^1(\Omega))$ such that $u(0, \cdot) = u_0$ and,

$$\forall v \in X, \quad \int_{\Omega} f(\cdot, \partial_t u) v dx + a_p(u, v) + \epsilon a_2(\partial_t u, v) = \int_{\Omega} g v dx, \quad t \in]0, T[\text{ a.e.}$$

If $\epsilon = 0$, a weak solution of (P_t) is any $u \in W^{1,p,2}(0, T; W_0^{1,p}(\Omega), L^2(\Omega))$ such that $u(0, \cdot) = u_0$ and,

$$\forall v \in X, \quad \int_{\Omega} f(\cdot, \partial_t u) v dx + a_p(u, v) = \int_{\Omega} g v dx, \quad t \in]0, T[\text{ a.e.}$$

Our main existence result is stated in the following theorem:

Theorem 1.2. Assume that $u_0 \in W_0^{1,p}(\Omega)$, then there exists a weak solution u in the sense of the definition (1.1). Moreover, $u \in C([0, T], W_0^{1,p}(\Omega))$ and satisfies the energy equality: for any $t > 0$

$$\int_{]0, t[\times \Omega} [f(\partial_t u) \partial_t u + \epsilon |\nabla \partial_t u|^2] dx ds + \frac{1}{p} \|u(t)\|_{W_0^{1,p}(\Omega)}^p = \int_{]0, t[\times \Omega} g \partial_t u dx ds + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p.$$

Remark 1.3. When $\epsilon > 0$ and $u_0 \in X$ one has that $u \in C([0, T], X)$.

Next, we are interested in the case of a stochastic perturbation of the Barenblatt equation in the Hilbert case: $f(\partial_t u) - \Delta u = 0$. This equation is equivalent to $\partial_t u = f^{-1} \Delta u$ and then, the stochastic perturbation would be $du = f^{-1} \Delta u dt + h dw$. Then, following J.U. KIM [19], B. EWALDÉ, M. PETCU AND R. TEMAM [14], G. VALLET [26] and G. VALLET, P. WITTBOLD [27], the stochastic version of the equation can be interpreted in the following sense: $\partial_t [u - \int_0^t h dw(s)] \in f^{-1} \Delta u$, where $\int_0^t h dw(s)$ denotes the Itô integration of h .

So, in Section 3, we consider Problem (S_t) . (Θ, \mathcal{F}, P) is a probability space and $w = \{w_t, \mathcal{F}_t, 0 \leq t \leq T\}$ is a standard adapted one dimensional continuous Brownian motion defined on (Θ, \mathcal{F}, P) such that $w_0 = 0$.

Regarding Problem (S_t) , we require the following conditions on f :

(f6) f is an increasing function such that f and f^{-1} are Lipschitz-continuous and $f(0) = 0$.

Concerning the data u_0 and h , we assume that

(H) $u_0 \in H_0^1(\Omega)$, $h \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ the space of predictable $L^2(]0, T[\times \Theta; H_0^1(\Omega))$ functions (cf. [22] p.28).

We define the notion of weak solution to (S_t) as follows:

Definition 1.4. Any function $u \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ such that $\partial_t [u - \int_0^t h dw] \in L^2(\Theta \times Q)$ is a solution to the stochastic problem (S_t) if $u(0, \cdot) = u_0$, and if for t almost everywhere in $(0, T)$ and any test function v of $H_0^1(\Omega)$ the following variational formulation holds:

$$\int_D f \left(\partial_t [u - \int_0^t h dw] \right) v dx + a_2(u, v) = 0.$$

We now state the main result about the solvability of (S_t) :

Theorem 1.5. *Under the assumptions (f6) and (H), there exists a unique solution u in the sense of the definition 1.4 to (S_t) .*

Moreover, u is an $H_0^1(\Omega)$ -valued process adapted to the filtration $(\mathcal{F}_t)_t$.

We derive an application to the multiplicative case in Theorem 3.2.

2 Proof of Theroem 1.2

In this section, we prove Theorem 1.2:

Proof of Theorem 1.2. We prove first the existence of a weak solution, when $u^0 \in X$ if $\epsilon > 0$ and in $W_0^{1,p}(\Omega)$ else, by a time-discretization argument; then, in the general case.

Consider u^0 a function in X (resp. $W_0^{1,p}(\Omega)$) if $\epsilon > 0$ (resp. $\epsilon = 0$) and set $B \stackrel{\text{def}}{=} B_{\Delta t, u^0, \epsilon}$ the operator

$$B : X \rightarrow X', v \mapsto f\left(\cdot, \frac{v - u^0}{\Delta t}\right) - \Delta_p v - \epsilon \Delta \left(\frac{v - u^0}{\Delta t}\right).$$

From the variational structure of B , B is associated to the energy:

$$J : X \rightarrow \mathbb{R}, v \mapsto \Delta t \int_{\Omega} \int_0^{\frac{v - u^0}{\Delta t}} f(\cdot, s) ds dx + \frac{1}{p} \int_{\Omega} |\nabla v|^p dx + \Delta t \frac{\epsilon}{2} \int_{\Omega} \left| \nabla \frac{v - u^0}{\Delta t} \right|^2 dx.$$

Then, we have the following result :

Lemma 2.1. *For any $g \in L^2(\Omega)$ and for any $\epsilon \geq 0$, there exists a unique $w \in X$ such that $Bw = g$ in the sense: $w \in X$ and,*

$$\forall v \in X, \int_{\Omega} f\left(\cdot, \frac{w - u^0}{\Delta t}\right) v dx + a_p(w, v) + \epsilon a_2\left(\frac{w - u^0}{\Delta t}, v\right) = \int_{\Omega} g v dx.$$

Proof. From the assumptions (f1) and (f2), J is Gâteaux-differentiable on the whole space X . Thanks to (f3) together with the strict monotonicity of p -laplace operator, J is strictly convex and from (f4) ((f5), resp.), J is coercive (i.e. $J(u) \rightarrow +\infty$ as $\|u\|_X \rightarrow +\infty$). Again from (f4) ((f5), resp.) and Fatou's Lemma, J is weakly lower semi-continuous in X . Therefore, there exists a unique global minimizer to J , we denote by w and which satisfied $J'(w)(v) = 0$ for any $v \in X$. \square

Then, iterating the application of Lemma 2.1, it is immediate to derive the following result:

Lemma 2.2. *Consider $(g^n) \subset L^2(\Omega)$. There exists a unique sequence $(u^n) \in X$ such that $u^0 = u_0$ and*

$$\forall v \in X, \int_{\Omega} f\left(\cdot, \frac{u^{n+1} - u^n}{\Delta t}\right) v dx + a_p(u^{n+1}, v) + \epsilon a_2\left(\frac{u^{n+1} - u^n}{\Delta t}, v\right) = \int_{\Omega} g^n v dx. \quad (2.1)$$

Setting $v = \frac{u^{n+1} - u^n}{\Delta t}$ as a test function in (2.1), convexity arguments, assumptions (f4) (resp. (f5) with $C_0 = -C < 0$) and Young's inequality (see P. LINDQVIST [20] in the annexe) yield

$$\begin{aligned} & -C_\epsilon \left\| \frac{u^{n+1} - u^n}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{1}{p\Delta t} \left[\|\nabla u^{n+1}\|_{L^p(\Omega)}^p - \|\nabla u^n\|_{L^p(\Omega)}^p + c(p) \|\nabla(u^{n+1} - u^n)\|_{L^p(\Omega)}^p \right] \\ & + \epsilon \left\| \nabla \frac{u^{n+1} - u^n}{\Delta t} \right\|_{L^2(\Omega)}^2 \leq C(b_2) \|g^n\|_{L^2(\Omega)}^2 + \frac{b_2}{2} \left\| \frac{u^{n+1} - u^n}{\Delta t} \right\|_{L^2(\Omega)}^2 + b_1, \end{aligned}$$

where $b_1 > 0$ is large enough (depending only on b_2 and $\|C_1\|_{L^2(\Omega)}$) and $c(p) > 0$ if $p \geq 2$, and 0 otherwise.

If $\epsilon > 0$, there exists $\delta > 0$ such that $\lambda_1(2, \Omega)\epsilon - C_\epsilon \geq \delta\epsilon$ and one claims $b_2 = \delta\epsilon$. Else, if $\epsilon = 0$, one claims $b_2 = -C_0 = C$. Then, setting such b_2 in the above expression, we obtain

Lemma 2.3.

$$\begin{aligned} \text{If } \epsilon > 0 : & \quad \Delta t \sum_{k=0}^{k=n} \left[c(p) \left\| \nabla \frac{u^{k+1} - u^k}{\sqrt[p]{\Delta t}} \right\|_{L^p(\Omega)}^p + \epsilon \frac{\delta}{2\lambda_1(2, \Omega)} \left\| \nabla \frac{u^{k+1} - u^k}{\Delta t} \right\|_{L^2(\Omega)}^2 \right] + \frac{1}{p} \|\nabla u^{n+1}\|_{L^p}^p \\ & \leq \Delta t C(b_2) \sum_{k=0}^{k=n} \|g^n\|_{L^2(\Omega)}^2 + \frac{1}{p} \|\nabla u_0\|_{L^p}^p + T b_1, \\ \text{If } \epsilon = 0 : & \quad \Delta t \sum_{k=0}^{k=n} \left[c(p) \left\| \nabla \frac{u^{k+1} - u^k}{\sqrt[p]{\Delta t}} \right\|_{L^p(\Omega)}^p + \frac{C}{2} \left\| \frac{u^{k+1} - u^k}{\Delta t} \right\|_{L^2(\Omega)}^2 \right] + \frac{1}{p} \|\nabla u^{n+1}\|_{L^p}^p \\ & \leq \Delta t C(b_2) \sum_{k=0}^{k=n} \|g^n\|_{L^2(\Omega)}^2 + \frac{1}{p} \|\nabla u_0\|_{L^p}^p + T b_1. \end{aligned}$$

Setting $g^k = \frac{1}{\Delta t} \int_{t_k}^{t_{k+1}} g(t, \cdot) dt$, we derive from the above lemma the following result:

Lemma 2.4. *There exists a positive constant $C = C(\|g\|_{L^2(Q)}, \|\nabla u_0\|_{L^p}, f)$, independent of ϵ , such that*

1. *If $\epsilon > 0$: $\sqrt{\epsilon} \|\partial_t \tilde{u}^{\Delta t}\|_{L^2(0, T; H_0^1(\Omega))} \leq C$, if $\epsilon = 0$: $\|\partial_t \tilde{u}^{\Delta t}\|_{L^2(Q)} \leq C$.*
2. *$\|u^{\Delta t}\|_{L^\infty(0, T; W_0^{1,p}(\Omega))} + \|\tilde{u}^{\Delta t}\|_{L^\infty(0, T; W_0^{1,p}(\Omega))} \leq C$.*
3. *$\sqrt{\epsilon} \left\| \frac{\tilde{u}^{\Delta t} - u^{\Delta t}}{\Delta t} \right\|_{L^2(0, T; H_0^1(\Omega))} + \sqrt{c(p)} \left\| \frac{\tilde{u}^{\Delta t} - u^{\Delta t}}{\sqrt[p]{\Delta t}} \right\|_{L^p(0, T; W_0^{1,p}(\Omega))} \leq C$.
In particular, if $\epsilon > 0$, $\|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{L^p(0, T; X)}$ goes to 0 as $\Delta t \rightarrow 0^+$.*
4. *$(f(\cdot, \partial_t \tilde{u}^{\Delta t}))$ and $(\frac{\tilde{u}^{\Delta t} - u^{\Delta t}}{\Delta t})$ are bounded in $L^2(Q)$, independently of Δt .*

To pass to the limit, we distinguish the following two cases : the case $\epsilon > 0$ and the case $\epsilon = 0$.

Let us start with the case $\epsilon > 0$.

In particular, $u_0 \in X$ and up to a subsequence, denoted in the same way:

- (i) there exists u_ϵ in $L^\infty(0, T, W_0^{1,p}(\Omega)) \cap H^1(0, T; H_0^1(\Omega))$ such that: $\partial_t \tilde{u}^{\Delta t}$ converges weakly to $\partial_t u_\epsilon$ in $L^2(0, T, H_0^1(\Omega))$, $\tilde{u}^{\Delta t}$ and $u^{\Delta t}$ converge *-weakly to u_ϵ in $L^\infty(0, T, W_0^{1,p}(\Omega))$, $\tilde{u}^{\Delta t}(t)$ converges weakly to $u_\epsilon(t)$ in X for any $t > 0$.

(ii) Note that, since $u_0 \in X$, one gets that $u_\epsilon \in L^\infty(0, T, X)$ and $u_\epsilon \in C^{0,1/2}([0, T], H_0^1(\Omega)) \cap C_w([0, T], X)$.

(iii) $\tilde{u}^{\Delta t} - u^{\Delta t}$ converges to 0 in $L^p(0, T, W_0^{1,p}(\Omega))$ and $L^2(0, T, H_0^1(\Omega))$ as $\Delta t \rightarrow 0^+$.

(iv) there exists χ in $L^2(Q)$, weak limit of $f(\cdot, \partial_t \tilde{u}^{\Delta t})$ as $\Delta t \rightarrow 0^+$.

For any $v \in X$,

$$\int_{\Omega} f(\cdot, \partial_t \tilde{u}^{\Delta t}) v dx + a_p(\tilde{u}^{\Delta t}, v) + \epsilon a_2(\partial_t \tilde{u}^{\Delta t}, v) = \int_{\Omega} g^{\Delta t} v dx + a_p(\tilde{u}^{\Delta t}, v) - a_p(u^{\Delta t}, v).$$

Thus, for $v = \tilde{u}^{\Delta t} - u_\epsilon$ and any positive t , we get

$$\begin{aligned} & \int_0^t \int_{\Omega} f(\cdot, \partial_t \tilde{u}^{\Delta t})(\tilde{u}^{\Delta t} - u_\epsilon) dx ds + \int_0^t a_p(\tilde{u}^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds + \epsilon \int_0^t a_2(\partial_t \tilde{u}^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds \\ &= \int_0^t \int_{\Omega} g^{\Delta t}(\tilde{u}^{\Delta t} - u_\epsilon) dx ds + \int_0^t a_p(\tilde{u}^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds - \int_0^t a_p(u^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds. \end{aligned}$$

$\tilde{u}^{\Delta t}$ is bounded in $L^\infty(0, T, W_0^{1,p}(\Omega))$ and $\partial_t \tilde{u}^{\Delta t}$ is bounded in $L^2(0, T, H_0^1(\Omega))$ with $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ since $p > \frac{2d}{d+1}$. Then, the theorem of Simon (Cf. Th. 4.1 in the Annexes) asserts that $\tilde{u}^{\Delta t} - u_\epsilon$ converges to 0 in $C^0([0, T], L^2(\Omega))$ as $\Delta t \rightarrow 0^+$

$$\int_0^t \int_{\Omega} f(\cdot, \partial_t \tilde{u}^{\Delta t})(\tilde{u}^{\Delta t} - u_\epsilon) dx ds \rightarrow 0 \quad \text{and} \quad \int_0^t \int_{\Omega} g^{\Delta t}(\tilde{u}^{\Delta t} - u_\epsilon) dx ds \rightarrow 0. \quad (2.2)$$

Then, setting $q = \frac{p}{p-1}$ and from J. SIMON [23] (2.11), we obtain that

$$\begin{aligned} & \int_0^t a_p(\tilde{u}^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds - \int_0^t a_p(u^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds \\ &= \int_0^t \int_{\Omega} [|\nabla \tilde{u}^{\Delta t}|^{p-2} \nabla \tilde{u}^{\Delta t} - |\nabla u^{\Delta t}|^{p-2} \nabla u^{\Delta t}] \nabla(\tilde{u}^{\Delta t} - u_\epsilon) ds \\ &\leq C \int_0^t \|\nabla \tilde{u}^{\Delta t}|^{p-2} \nabla \tilde{u}^{\Delta t} - |\nabla u^{\Delta t}|^{p-2} \nabla u^{\Delta t}\|_{L^q(\Omega)} \|\tilde{u}^{\Delta t} - u_\epsilon\|_{W_0^{1,p}(\Omega)} ds \\ &\leq C \int_0^t \|\nabla \tilde{u}^{\Delta t}|^{p-2} \nabla \tilde{u}^{\Delta t} - |\nabla u^{\Delta t}|^{p-2} \nabla u^{\Delta t}\|_{L^q(\Omega)} ds \\ &\leq C \begin{cases} \int_0^t \max(\|\tilde{u}^{\Delta t}\|_{W_0^{1,p}(\Omega)}^{p-2}, \|u^{\Delta t}\|_{W_0^{1,p}(\Omega)}^{p-2}) \|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{W_0^{1,p}(\Omega)} ds & \text{if } p \geq 2 \\ \int_0^t \|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{W_0^{1,p}(\Omega)}^{p-1} ds & \text{if } p \leq 2 \end{cases} \\ &\leq C \|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{L^p(0, T, W_0^{1,p}(\Omega))} \rightarrow 0 \text{ thanks to (3) in lemma 2.4.} \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^t a_2(\partial_t \tilde{u}^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds &= \int_0^t a_2(\partial_t \tilde{u}^{\Delta t} - \partial_t u_\epsilon, \tilde{u}^{\Delta t} - u_\epsilon) ds + \int_0^t a_2(\partial_t u_\epsilon, \tilde{u}^{\Delta t} - u_\epsilon) ds \\ &= \frac{1}{2} a_2(\tilde{u}^{\Delta t} - u_\epsilon, \tilde{u}^{\Delta t} - u_\epsilon) + \int_0^t a_2(\partial_t u_\epsilon, \tilde{u}^{\Delta t} - u_\epsilon) ds, \end{aligned}$$

where $\int_0^t a_2(\partial_t u_\epsilon, \tilde{u}^{\Delta t} - u_\epsilon) ds$ converges to 0 as $\Delta t \rightarrow 0^+$. Therefore, we obtain that

$$\limsup_{\Delta t \rightarrow 0} \int_0^t a_p(\tilde{u}^{\Delta t}, \tilde{u}^{\Delta t} - u_\epsilon) ds \leq 0,$$

and from the S^+ property satisfied by $-\Delta_p$, $\tilde{u}^{\Delta t}$ and consequently $u^{\Delta t}$, converge both to u_ϵ in $L^p(0, T, W_0^{1,p}(\Omega))$ (see for instance DINCA-JEBELEAN-MAWHIN [13] Th.10) .

Remark that, for any $t \in]t_n, t_{n+1}[$

$$\int_{\Omega} f(\cdot, \partial_t \tilde{u}^{\Delta t}) \partial_t \tilde{u}^{\Delta t} dx + \frac{1}{p \Delta t} \|u^{n+1}\|_{W_0^{1,p}(\Omega)}^p + \epsilon \|\partial_t \tilde{u}^{\Delta t}\|_{H_0^1(\Omega)}^2 \leq \int_{\Omega} g^{\Delta t} \partial_t \tilde{u}^{\Delta t} dx + \frac{1}{p \Delta t} \|u^n\|_{W_0^{1,p}(\Omega)}^p.$$

Then, summing over n , we get

$$\int_0^T (A_\epsilon \partial_t \tilde{u}^{\Delta t}, \partial_t \tilde{u}^{\Delta t}) dt + \frac{1}{p} \|\tilde{u}^{\Delta t}(T)\|_{W_0^{1,p}(\Omega)}^p \leq \int_Q g^{\Delta t} \partial_t \tilde{u}^{\Delta t} dx + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p,$$

and passing to the limit as $\Delta t \rightarrow 0^+$,

$$\limsup_{\Delta t \rightarrow 0^+} \int_0^T (A_\epsilon \partial_t \tilde{u}^{\Delta t}, \partial_t \tilde{u}^{\Delta t}) dt + \frac{1}{p} \|u_\epsilon(T)\|_{W_0^{1,p}(\Omega)}^p \leq \int_Q g \partial_t u_\epsilon dx + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p.$$

Moreover, at the limit, one has, for any $v \in X$,

$$\int_{\Omega} \chi v dx + a_p(u_\epsilon, v) + \epsilon a_2(\partial_t u_\epsilon, v) = \int_{\Omega} g v dx.$$

For any $\delta > 0$, denote by v_δ the solution of the following mean value problem: $\delta v'_\delta + v_\delta = u_\epsilon$ when $t \geq s$ and $v_\delta(s) = u_\epsilon(s)$.

Since $u_\epsilon \in H^1(s, T, H_0^1(\Omega)) \cap C_w([s, T], W_0^{1,p}(\Omega))$, as δ goes to 0^+ , v_δ converges weakly to u_ϵ in $H^1(s, T; H_0^1(\Omega))$ and $v_\delta(t)$ converges weakly to $u_\epsilon(t)$ in $W_0^{1,p}(\Omega)$ for any $t \geq 0$.

Then, using the test-function $u_\epsilon - v_\delta$ and the monotonicity of a_p yield

$$\int_{\Omega} \chi v'_\delta dx + \frac{1}{p} \frac{d}{dt} \|v_\delta\|_{W_0^{1,p}(\Omega)}^p + \epsilon a_2(\partial_t u_\epsilon, v'_\delta) \leq \int_{\Omega} g v'_\delta dx.$$

Integrating in time and passing to the limit as $\delta \rightarrow 0^+$, we get

$$\begin{aligned} & \int_s^t \int_{\Omega} \chi \partial_t u_\epsilon dx d\sigma + \frac{1}{p} \limsup_{\delta \rightarrow 0^+} \|v_\delta(t)\|_{W_0^{1,p}(\Omega)}^p + \epsilon \int_s^t a_2(\partial_t u_\epsilon, \partial_t u_\epsilon) d\sigma \\ & \leq \int_s^t \int_{\Omega} g \partial_t u_\epsilon dx d\sigma + \frac{1}{p} \|u_\epsilon(s)\|_{W_0^{1,p}(\Omega)}^p, \end{aligned}$$

and, thanks to the weak convergence,

$$\int_s^t \int_{\Omega} \chi \partial_t u_\epsilon dx d\sigma + \frac{1}{p} \|u_\epsilon(t)\|_{W_0^{1,p}(\Omega)}^p + \epsilon \int_s^t a_2(\partial_t u_\epsilon, \partial_t u_\epsilon) d\sigma \leq \int_s^t \int_{\Omega} g \partial_t u_\epsilon dx d\sigma + \frac{1}{p} \|u_\epsilon(s)\|_{W_0^{1,p}(\Omega)}^p \quad (2.3)$$

Thus, we get that $\limsup_{t \rightarrow s^+} \|u_\epsilon(t)\|_{W_0^{1,p}(\Omega)} \leq \|u_\epsilon(s)\|_{W_0^{1,p}(\Omega)}$ and that $t \mapsto u_\epsilon(t)$ is right-continuous in $W_0^{1,p}(\Omega)$ since it is $C_w([0, T], W_0^{1,p}(\Omega))$.

Then, for any h and t such that $0 \leq t < t+h \leq T$,

$$\int_{\Omega} \chi \frac{u_\epsilon(t+h) - u_\epsilon(t)}{h} dx + a_p(u_\epsilon(t), \frac{u_\epsilon(t+h) - u_\epsilon(t)}{h}) + \epsilon a_2(\partial_t u_\epsilon(t), \frac{u_\epsilon(t+h) - u_\epsilon(t)}{h}) = \int_{\Omega} g(t) \frac{u_\epsilon(t+h) - u_\epsilon(t)}{h} dx,$$

and, integrating in time from 0 to t , we get

$$\begin{aligned} \int_0^t \int_{\Omega} g(s) \frac{u_\epsilon(s+h) - u_\epsilon(s)}{h} dx ds & \leq \int_0^t \left[\int_{\Omega} \chi \frac{u_\epsilon(s+h) - u_\epsilon(s)}{h} dx + \epsilon a_2(\partial_t u_\epsilon(s), \frac{u_\epsilon(s+h) - u_\epsilon(s)}{h}) \right] ds \\ & \quad + \frac{1}{hp} \left[\int_t^{t+h} \|u_\epsilon(s)\|_{W_0^{1,p}(\Omega)}^p ds - \int_0^h \|u_\epsilon(s)\|_{W_0^{1,p}(\Omega)}^p ds \right]. \end{aligned}$$

$\frac{u_\epsilon(\cdot+h)-u_\epsilon}{h}$ converges to $\partial_t u_\epsilon$ in $L^2(0, T, H_0^1(\Omega))$ and u_ϵ is right-continuous in $W_0^{1,p}(\Omega)$, then one gets that

$$\int_{]0,t[\times \Omega} g \partial_t u_\epsilon \, dx dt \leq \int_{]0,t[\times \Omega} \chi \partial_t u_\epsilon + \epsilon |\nabla \partial_t u_\epsilon(t)|^2 \, dx dt + \frac{1}{p} [\|u_\epsilon(t)\|_{W_0^{1,p}(\Omega)}^p - \|u_0\|_{W_0^{1,p}(\Omega)}^p]. \quad (2.4)$$

Since t and T are arbitrary, we can assume that it is true for $t = T$ and

$$\limsup_{\Delta t \rightarrow 0^+} \int_0^T (A_\epsilon \partial_t \tilde{u}^{\Delta t}, \partial_t \tilde{u}^{\Delta t}) \, dt \leq \int_0^T [\int_\Omega \chi \partial_t u_\epsilon \, dx + \epsilon a_2(\partial_t u_\epsilon, \partial_t u_\epsilon)] dt.$$

Then, $A_\epsilon \partial_t u_\epsilon = \chi - \epsilon \Delta \partial_t u_\epsilon$, *i.e.* $f(\cdot, \partial_t u_\epsilon) = \chi$ thanks to a monotonicity argument from which it follows that u_ϵ is a weak solution to (P_t) .

Then, (2.3) with $s = 0$ and (2.4) yield the energy equality and the continuity of $t \mapsto \|u_\epsilon\|_{W_0^{1,p}(\Omega)}$. Since $u_\epsilon \in C_w([0, T], W_0^{1,p}(\Omega))$, one gets that $u_\epsilon \in C([0, T], W_0^{1,p}(\Omega))$. This achieves the proof in case where $\epsilon > 0$ and $u_0 \in X$, *i.e.* the general case for $\epsilon > 0$ when $p \geq 2$.

Let us now consider the case : $\frac{2d}{d+2} < p < 2$, $\epsilon > 0$ and $u_0 \in W_0^{1,p}(\Omega)$.

Denote by $(u_0^n) \subset X$ a sequence that converges to u_0 in $W_0^{1,p}(\Omega)$. Thanks to Theorem 1.2, there exists a sequence $(u^n) \subset C([0, T], X)$, of solutions of (P_t) in the sense of Definition 1.1, for the initial condition u_0^n .

Since $p < 2$, $\partial_t u^n$ is a test-function and one gets that (u^n) is bounded in $W^{1,\infty,2}(0, T, W_0^{1,p}(\Omega), H_0^1(\Omega))$ and $(f(\cdot, \partial_t u^n))$ is bounded in $L^2(Q)$. Denote by u a limit-point of (u^n) for the weak (weak-*) convergence in $W^{1,\infty,2}(0, T, W_0^{1,p}(\Omega), H_0^1(\Omega))$, $v^n(\cdot, t) = u_0^n + \int_0^t \partial_t u(\cdot, s) ds$ and χ a limit-point of $(f(\cdot, \partial_t u^n))$ for the weak convergence in $L^2(Q)$.

By construction, $v^n \in H^1(0, T, H_0^1(\Omega))$ and $\|u - v^n\|_{L^\infty(0, T, W_0^{1,p}(\Omega))} = \|u_0 - u_0^n\|_{W_0^{1,p}(\Omega)}$ goes to 0 as n goes to $+\infty$. Moreover, $\|u^n - v^n\|_{L^\infty(0, T, H_0^1(\Omega))} \leq \int_0^T \|\partial_t(u^n - u)\|_{H_0^1(\Omega)} dt$. Thus, $u^n - v^n$ converges weakly to 0 in $L^2(0, T, H_0^1(\Omega))$

Thus, taking $v = u^n - v^n$ as a test-function, and for any $t > 0$, we get

$$\int_0^t \int_\Omega [f(\cdot, \partial_t u^n)(u^n - v^n) dx ds + a_p(u^n, u^n - v^n) + \epsilon a_2(\partial_t u^n, u^n - v^n)] ds = \int_0^t \int_\Omega g(u^n - v^n) dx ds.$$

Ascoli's theorem asserts that $u^n - u$ converges to 0 in $C([0, T], L^2(\Omega))$. Thus $u^n - v^n$ converges to 0 in $L^\infty(0, T, L^2(\Omega))$ and

$$\int_0^t \int_\Omega f(\cdot, \partial_t u^n)(u^n - v^n) dx ds \rightarrow 0 \quad \text{and} \quad \int_0^t \int_\Omega g(u^n - v^n) dx ds \rightarrow 0. \quad (2.5)$$

Moreover,

$$\begin{aligned} \int_0^t a_2(\partial_t u^n, u^n - v^n) ds &= \int_0^t a_2(\partial_t(u^n - v^n), u^n - v^n) ds + \int_0^t a_2(\partial_t v^n, u^n - v^n) ds \\ &= \frac{1}{2} a_2(u^n - v^n, u^n - v^n) + \int_0^t a_2(\partial_t u, u^n - v^n) ds, \\ \text{and} \quad \limsup_{n \rightarrow \infty} \int_0^t a_p(u^n, u^n - v^n) ds &\leq 0. \end{aligned}$$

Thus, $\limsup_{n \rightarrow \infty} \int_0^t a_p(u^n, u^n - u) ds \leq 0$ and u^n converges to u in $L^p(0, T, W_0^{1,p}(\Omega))$.

Since,

$$\int_Q f(\cdot, \partial_t u^n) \partial_t u^n \, dx dt + \frac{1}{p} \|u^n(T)\|_{W_0^{1,p}(\Omega)}^p + \epsilon \int_0^T \|\partial_t u^n\|_{H_0^1(\Omega)}^2 ds = \int_Q g \partial_t u^n \, dx dt + \frac{1}{p} \|u_0^n\|_{W_0^{1,p}(\Omega)}^p,$$

$$\text{then, } \int_0^T (A_\epsilon \partial_t u^n, \partial_t u^n) \, dt + \frac{1}{p} \|u^n(T)\|_{W_0^{1,p}(\Omega)}^p \leq \int_Q g \partial_t u^n \, dx + \frac{1}{p} \|u_0^n\|_{W_0^{1,p}(\Omega)}^p,$$

and passing to the limit as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \int_0^T (A_\epsilon \partial_t u^n, \partial_t u^n) \, dt + \frac{1}{p} \|u(T)\|_{W_0^{1,p}(\Omega)}^p \leq \int_Q g \partial_t u \, dx + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p.$$

Moreover, at the limit, one has, for any $v \in H_0^1(\Omega)$,

$$\int_\Omega \chi v \, dx + a_p(u, v) + \epsilon a_2(\partial_t u, v) = \int_\Omega g v \, dx.$$

Set $v = \partial_t u^n$ above. Since, $\partial_t u^n$ converges weakly to $\partial_t u$ in $L^2(0, T, H_0^1(\Omega))$, one gets that

$$\int_\Omega \chi \partial_t u \, dx + a_p(u, \partial_t u) + \epsilon a_2(\partial_t u, \partial_t u) = \int_\Omega g \partial_t u \, dx.$$

Integrating in time[‡], we get

$$\int_Q \chi \partial_t u \, dx d\sigma + \frac{1}{p} \|u(T)\|_{W_0^{1,p}(\Omega)}^p + \epsilon \int_0^T a_2(\partial_t u, \partial_t u) d\sigma = \int_Q g \partial_t u \, dx d\sigma + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p. \quad (2.6)$$

Thus,

$$\limsup_{\Delta t \rightarrow 0^+} \int_0^T (A_\epsilon \partial_t \tilde{u}^{\Delta t}, \partial_t \tilde{u}^{\Delta t}) \, dt \leq \int_0^T [\int_\Omega \chi \partial_t u \, dx + \epsilon a_2(\partial_t u, \partial_t u)] dt,$$

and achieves the proof in case where $\epsilon > 0$ since $u \in W^{1,p,2}(0, T, W_0^{1,p}(\Omega), H_0^1(\Omega))$ yields the regularity and the energy property.

Let us now consider the case $\epsilon = 0$. There, u_0 is in $W_0^{1,p}(\Omega)$ and using similar arguments as in the previous case, we get

(ibis) there exists u in $L^\infty(0, T, W_0^{1,p}(\Omega)) \cap H^1(0, T; L^2(\Omega))$ such that: $\partial_t \tilde{u}^{\Delta t}$ converges weakly to $\partial_t u$ in $L^2(0, T, L^2(\Omega))$, $\tilde{u}^{\Delta t}$ and $u^{\Delta t}$ converge *-weakly to u in $L^\infty(0, T, W_0^{1,p}(\Omega))$, $\tilde{u}^{\Delta t}(t)$ converges weakly to $u(t)$ in $W_0^{1,p}(\Omega)$ for any t .

(iibis) $\tilde{u}^{\Delta t} - u^{\Delta t}$ converges to 0 in $L^2(Q)$.

(iiibis) there exists χ in $L^2(Q)$, weak limit in the same space of $f(\cdot, \partial_t \tilde{u}^{\Delta t})$.

Multiplying the equation by $v = u^{\Delta t} - u$ gives, for any positive t ,

$$\int_0^t \int_\Omega f(\cdot, \partial_t \tilde{u}^{\Delta t})(u^{\Delta t} - u) \, dx ds + \int_0^t a_p(u^{\Delta t}, u^{\Delta t} - u) \, ds = \int_0^t \int_\Omega g^{\Delta t}(u^{\Delta t} - u) \, dx ds.$$

[‡]This is possible since $u \in W^{1,p}(0, T, W_0^{1,p}(\Omega))$.

$\tilde{u}^{\Delta t}$ is bounded in $L^\infty(0, T, W_0^{1,p}(\Omega))$, $\partial_t \tilde{u}^{\Delta t}$ bounded in $L^2(0, T, L^2(\Omega))$ and $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ ($p > \frac{2d}{d+2}$), then Simon's theorem (Cf. Th. 4.1 in the Annexes) implies that $\tilde{u}^{\Delta t} - u$ converges to 0 in $C^0([0, T], L^2(\Omega))$.

Then, since $\tilde{u}^{\Delta t} - u^{\Delta t}$ converges to 0 in $L^2(Q)$,

$$\int_0^t \int_\Omega f(\cdot, \partial_t \tilde{u}^{\Delta t})(u^{\Delta t} - u) dx ds \rightarrow 0 \quad \text{and} \quad \int_0^t \int_\Omega g^{\Delta t}(u^{\Delta t} - u) dx ds \rightarrow 0. \quad (2.7)$$

Therefore, one gets that

$$\limsup_{\Delta t \rightarrow 0} \int_0^t a_p(u^{\Delta t}, u^{\Delta t} - u) ds \leq 0,$$

and $u^{\Delta t}$ converges to u in $L^p(0, T, W_0^{1,p}(\Omega))$.

Following the proof of the previous case when $\epsilon > 0$, we have that

$$\int_Q f(\cdot, \partial_t \tilde{u}^{\Delta t}) \partial_t \tilde{u}^{\Delta t} dx dt + \frac{1}{p} \|\tilde{u}^{\Delta t}(T)\|_{W_0^{1,p}(\Omega)}^p \leq \int_Q g^\Delta \partial_t \tilde{u}^{\Delta t} dx + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p,$$

and at the limit,

$$\limsup_{\Delta t} \int_Q f(\cdot, \partial_t \tilde{u}^{\Delta t}) \partial_t \tilde{u}^{\Delta t} dx dt + \frac{1}{p} \|u(T)\|_{W_0^{1,p}(\Omega)}^p \leq \int_Q g \partial_t u dx + \frac{1}{p} \|u_0\|_{W_0^{1,p}(\Omega)}^p,$$

Moreover, at the limit, one has, for any $v \in X$,

$$\int_\Omega \chi v dx + a_p(u, v) = \int_\Omega g v dx.$$

For any $\delta > 0$, denote by v_δ the solution of the mean value problem : $\delta v'_\delta + v_\delta = u$ when $t \geq s$ and $v_\delta(s) = u(s)$.

Since $u \in H^1(s, T, L^2(\Omega)) \cap C_w([s, T], W_0^{1,p}(\Omega))$, as $\delta \rightarrow 0^+$, v_δ converges to u in $H^1(s, T; L^2(\Omega))$ and $v_\delta(t)$ converges weakly to $u(t)$ in $W_0^{1,p}(\Omega)$ for any $t > 0$.

Then, the test-function $u - v_\delta$ and the monotonicity of a_p yield $\int_\Omega \chi v'_\delta dx + \frac{1}{p} \frac{d}{dt} \|v_\delta\|_{W_0^{1,p}(\Omega)}^p = \int_\Omega g v'_\delta dx$. Then, as above, one can prove that $\limsup_{t \rightarrow s^+} \|u(t)\|_{W_0^{1,p}(\Omega)} \leq \|u(s)\|_{W_0^{1,p}(\Omega)}$ and that u is right-continuous in $W_0^{1,p}(\Omega)$.

Then, similarly as in the case $\epsilon > 0$, one has that

$$\int_Q g \partial_t u dx dt \leq \int_Q \chi \partial_t u + \frac{1}{p} [\|u(T)\|_{W_0^{1,p}(\Omega)}^p - \|u_0\|_{W_0^{1,p}(\Omega)}^p].$$

Therefore, $\limsup_{\Delta t} \int_Q f(\cdot, \partial_t \tilde{u}^{\Delta t}) \partial_t \tilde{u}^{\Delta t} dx dt \leq \int_Q \chi \partial_t u dx dt$ and $f(\cdot, \partial_t u_\epsilon) = \chi$ since, when $\epsilon = 0$, f is nondecreasing.

Finally, we get in this case the energy equality and the continuity of u in $W_0^{1,p}(\Omega)$ as in the previous case. The proof of Theorem 1.2 is now complete \square

3 Proof of Theorem 1.5

We first assume that $h \in \mathcal{N}_w^2(0, T, H_0^1(\Omega) \cap H^2(\Omega))$. Then, [22]-Lemma 2.4.1 p. 35, yields $\Delta \int_0^t h dw(s) = \int_0^t \Delta h dw(s)$ a.s. in Θ as an element of $\mathcal{N}_w^2(0, T, L^2(\Omega))$.

Hence, the stochastic perturbation of the Barenblatt equation would be: $U(0, \cdot) = u_0$ and

$$f(\partial_t U) - \Delta U = \int_0^t \Delta h dw(s) = g \quad \text{where } U = u - \int_0^t h dw(s). \quad (3.1)$$

Remarking that U is the solution to $\partial_t U - \Delta U = g + \partial_t U - f(\partial_t U) \in L^2(Q)$ with $u_0 \in H_0^1(\Omega)$, thanks to the regularity of the solution of the heat equation (Cf. lemma 4.2), we get that U is unique and that it is an element of $H^1(Q) \cap C([0, T], H_0^1(\Omega))$. Moreover, using (4.1), if U and \hat{U} are the corresponding solutions to g and \hat{g} with the same initial condition, for any $t \in]0, T[$ one has that

$$\int_{]0, t[\times \Omega} [f(\partial_t U) - f(\partial_t \hat{U})] \partial_t [U - \hat{U}] dx ds + \frac{1}{2} \|U - \hat{U}\|_{H_0^1(\Omega)}^2(t) \leq \int_{]0, t[\times \Omega} [g - \hat{g}] \partial_t [U - \hat{U}] dx ds \quad (3.2)$$

Since f is a bi-Lipschitz continuous function, the application $\Psi : g \in L^2(0, T, L^2(\Omega)) \mapsto U(t) \in H_0^1(\Omega)$ is Lipschitz continuous.

Thus, since $g = \int_0^t \Delta h dw(s)$ is \mathcal{F}_t -measurable, we get that $U(t)$ is \mathcal{F}_t -measurable, *i.e.* U is adapted to the filtration. Finally, since a.s. $U \in C([0, T], H_0^1(\Omega))$, it is predictable and $U \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ (See G. DA PRATO, J. ZACZYK [10] Prop. 3.6 (i) p. 76 for adapted processes in $C_w([0, T], H_0^1(\Omega))$)

We conclude the proof in the case where $h \in \mathcal{N}_w^2(0, T, H_0^1(\Omega) \cap H^2(\Omega))$ by remarking that it is the same for $u = U + \int_0^t h dw(s)$.

Remark 3.1. Since $\int_0^t \Delta h dw$ is a square integrable continuous martingale ([10] Th. 4.12 p. 101), we get that ([10] Th. 3.8 (ii) p. 78)

$$\mathbb{E} \left[\sup_{t \in (0, T)} \left\| \int_0^t \Delta h dw \right\|_{L^2(\Omega)}^2 \right] \leq 4 \mathbb{E} \left\| \int_0^T \Delta h dw \right\|_{L^2(\Omega)}^2.$$

Moreover, almost surely $U \in C([0, T], H_0^1(\Omega))$ and, thanks to (4.1) applied to Problem $\partial_t U - \Delta U = g - f(\partial_t U) + \partial_t U$, we get that $\|U(t)\|_{H_0^1(\Omega)}^2 \leq \|u_0\|_{H_0^1(\Omega)}^2 + c \int_Q |\int_0^t \Delta h dw|^2 dx dt \leq c + c \sup_{t \in (0, T)} \left\| \int_0^t \Delta h dw \right\|_{L^2(\Omega)}^2$.

Then, Lebesgue theorem asserts that $t \mapsto \mathbb{E} \|U(t)\|_{H_0^1(\Omega)}^2$ is a continuous function.

Before dealing with the general case, we prove the following stability result :

Proposition 3.1. *Consider h and \hat{h} in $\mathcal{N}_w^2(H_0^1(\Omega) \cap H^2(\Omega))$, $u_0, \hat{u}_0 \in H_0^1(\Omega)$ and u, \hat{u} the associated solutions with $U = u - \int_0^t h dw$, $\hat{U} = \hat{u} - \int_0^t \hat{h} dw$. Then, for any t ,*

$$\begin{aligned} & \mathbb{E} \int_{]0, t[\times \Omega} (\partial_t (U - \hat{U}))^2 dx ds + \frac{1}{2} \mathbb{E} \|\nabla(u - \hat{u})(t)\|^2 \leq \\ & \leq \frac{1}{2} \mathbb{E} \|\nabla(u_0 - \hat{u}_0)\|^2 + \mathbb{E} \int_{]0, t[\times \Omega} |\nabla(h - \hat{h})|^2 dx ds. \end{aligned} \quad (3.3)$$

Proof. For all v in $H_0^1(\Omega)$, $\int_D [f(\partial_t U) - f(\partial_t \hat{U})]v dx + a(u - \hat{u}, v) = 0$.

Then, by taking the test function $v = \frac{(U - \hat{U})(t) - (U - \hat{U})(t - \Delta t)}{\Delta t}$, one gets

$$\begin{aligned} & \int_{\Omega} [f(\partial_t U) - f(\partial_t \hat{U})] \frac{(U - \hat{U})(t) - (U - \hat{U})(t - \Delta t)}{\Delta t} dx - \frac{1}{\Delta t} a(u(t) - \hat{u}(t), \int_{t-\Delta t}^t (h - \hat{h}) dw) \\ & + \frac{1}{\Delta t} a(u(t) - \hat{u}(t), u(t) - \hat{u}(t) - u(t - \Delta t) - \hat{u}(t - \Delta t)) = 0. \end{aligned}$$

Then, by noticing that

$$\begin{aligned} & a(u - \hat{u}, u - \hat{u} - (u - \hat{u})(t - \Delta t)) \\ & = \frac{1}{2\Delta t} [\|\nabla(u - \hat{u})\|^2 - \|\nabla(u - \hat{u})(t - \Delta t)\|^2 + \|\nabla[(u - \hat{u}) - (u - \hat{u})(t - \Delta t)]\|^2], \end{aligned}$$

we obtain

$$\begin{aligned} 0 & = \int_{\Omega} [f(\partial_t U) - f(\partial_t \hat{U})] \frac{(U - \hat{U})(t) - (U - \hat{U})(t - \Delta t)}{\Delta t} dx \\ & + \frac{1}{2\Delta t} [\|\nabla(u - \hat{u})(t)\|^2 - \|\nabla(u - \hat{u})(t - \Delta t)\|^2 + \|\nabla[(u - \hat{u})(t) - (u - \hat{u})(t - \Delta t)]\|^2] \\ & - \frac{1}{\Delta t} a\left((u - \hat{u})(t) - (u - \hat{u})(t - \Delta t), \int_{t-\Delta t}^t (h - \hat{h}) dw\right) \\ & - \frac{1}{\Delta t} a\left((u - \hat{u})(t - \Delta t), \int_{t-\Delta t}^t (h - \hat{h}) dw\right). \end{aligned}$$

And,

$$\begin{aligned} 0 & = \int_{\Omega} [f(\partial_t U) - f(\partial_t \hat{U})] \frac{(U - \hat{U})(t) - (U - \hat{U})(t - \Delta t)}{\Delta t} dx \\ & + \frac{1}{2\Delta t} [\|\nabla(u - \hat{u})(t)\|^2 - \|\nabla(u - \hat{u})(t - \Delta t)\|^2] \\ & - \frac{1}{4\Delta t} [\|\nabla[(u - \hat{u})(t) - (u - \hat{u})(t - \Delta t)]\|^2 - 4\|\nabla \int_{t-\Delta t}^t (h - \hat{h}) dw\|^2 \\ & + \|\nabla[(u - \hat{u})(t) - (u - \hat{u})(t - \Delta t) - 2 \int_{t-\Delta t}^t (h - \hat{h}) dw]\|^2] \\ & - \frac{1}{\Delta t} a\left((u - \hat{u})(t - \Delta t), \int_{t-\Delta t}^t (h - \hat{h}) dw\right) \end{aligned}$$

Then, by taking the expectation and the integral from Δt to T , and using the properties of the brownian motion, one gets

$$\begin{aligned} & \int_{\Delta t}^T \mathbb{E} \int_{\Omega} [f(\partial_t U) - f(\partial_t \hat{U})] \frac{(U - \hat{U})(t) - (U - \hat{U})(t - \Delta t)}{\Delta t} dx dt \\ & + \frac{1}{2\Delta t} \int_{\Delta t}^T \mathbb{E} [\|\nabla(u - \hat{u})(t)\|^2 - \|\nabla(u - \hat{u})(t - \Delta t)\|^2] dt \\ & \leq \int_{\Delta t}^T \mathbb{E} \frac{1}{\Delta t} \|\nabla \int_{t-\Delta t}^t (h - \hat{h}) dw\|^2 dt = \int_{\Delta t}^T \mathbb{E} \frac{1}{\Delta t} \int_{t-\Delta t}^t \|\nabla(h - \hat{h})\|^2 ds dt. \end{aligned}$$

i. e.

$$\begin{aligned} \mathbb{E} \int_{\Delta t}^T \int_{\Omega} [f(\partial_t U) - f(\partial_t \hat{U})] \frac{(U - \hat{U})(t) - (U - \hat{U})(t - \Delta t)}{\Delta t} dx dt & + \frac{1}{2\Delta t} \mathbb{E} \int_{T-\Delta t}^T \|\nabla(u - \hat{u})(t)\|^2 dt \\ & \leq \frac{1}{2\Delta t} \mathbb{E} \int_0^{\Delta t} \|\nabla(u - \hat{u})(t)\|^2 dt + \frac{1}{\Delta t} \mathbb{E} \int_{\Delta t}^T \int_{t-\Delta t}^t \|\nabla(h - \hat{h})\|^2 ds dt. \end{aligned}$$

And, by passing to the limit on Δt ,

$$\begin{aligned} \mathbb{E} \int_Q [f(\partial_t U) - f(\partial_t \hat{U})] \partial_t (U - \hat{U}) dx dt + \frac{1}{2} \mathbb{E} \|\nabla(u - \hat{u})(T)\|^2 &\leq \\ &\leq \frac{1}{2} \|\nabla(u_0 - \hat{u}_0)\|^2 + \mathbb{E} \int_Q |\nabla(h - \hat{h})|^2 dx dt. \end{aligned}$$

Since f is bi-Lipschitz and T is arbitrary, (3.3) holds and the proof of Proposition 3.1 is now complete. \square

Now we prove Theorem 1.5 in the case where $h \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$. Let us prove first the uniqueness of the solution: denote by u and \hat{u} two possible solutions, for the same initial condition u_0 and the same right-hand side term h . Then, one has, for any $v \in H_0^1(\Omega)$ and a.s., that

$$\int_Q [f(\partial_t U) - f(\partial_t \hat{U})] v + \nabla[u - \hat{u}] \nabla v dx dt = 0, \text{ i.e. } \int_Q [f(\partial_t U) - f(\partial_t \hat{U})] v + \nabla[U - \hat{U}] \nabla v dx dt = 0.$$

This means that $W := U - \hat{U}$ is the solution of the heat equation

$$\partial_t W - \Delta W = \partial_t (U - \hat{U}) - f(\partial_t U) + f(\partial_t \hat{U}) \in L^2(Q), \quad W(0, \cdot) = 0.$$

Thanks to (4.1) and to the monotonicity of f , the uniqueness of the solution follows. Now we show the existence of a solution: let us denote by $(h_n)_n \subset \mathcal{N}_w^2(0, T, H_0^1(\Omega) \cap H^2(\Omega))$ a sequence that converges to h in $\mathcal{N}_w^2(0, T, H_0^1(\Omega))$. From the first step of the proof, there exists (u_n) a sequence of solution of our problem in the sense of the definition 1.4.

Since $(h_n)_n$ is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H_0^1(\Omega))$, thanks to (3.3), (u_n) is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H_0^1(\Omega))$ as well as $(\partial_t[u_n - \int_0^t h_n dw])$ in $L^2(\Omega \times Q)$.

Moreover, for any t , $(u_n(t))$ is a Cauchy sequence in $L^2(\Theta, H_0^1(\Omega))$.

Thus, the limit u is a solution in the sense of Definition 1.4 and u is a $H_0^1(\Omega)$ -valued process, adapted to the filtration and (3.3) still holds. \square

3.1 The multiplicative case

Assume in this section that $H : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is a Lipschitz-continuous mapping. Then $H(h) \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ is $h \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ ([22] lemma 2.41 p.35), and the result of this section is

Theorem 3.2. *There exists a unique solution of Problem $f\left(\partial_t[u - \int_0^t H(u)dw]\right) - \Delta u = 0$ in $\Omega \times (0, T) \times \Theta$, $u(0, \cdot) = u_0$ in the sense of Definition 1.4.*

Denote by $\Phi : \mathcal{N}_w^2(0, T, H_0^1(\Omega)) \rightarrow \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ the map defined for any $h \in \mathcal{N}_w^2(0, T, H_0^1(\Omega))$ by $\Phi(h) = u$ where u is the solution of the Barenblatt's problem $f\left(\partial_t[u - \int_0^t H(h)dw]\right) - \Delta u = 0$ in $\Omega \times (0, T) \times \Theta$, $u(0, \cdot) = u_0$ in the sense of Definition 1.4.

Then, for any positive α , (3.3) yields

$$\begin{aligned} \int_0^T e^{-\alpha t} \mathbb{E} \|\Psi(h) - \Psi(\hat{h})(t)\|_{H_0^1(\Omega)}^2 dt &\leq 2 \int_0^T e^{-\alpha t} \int_0^t \mathbb{E} \| [H(h) - H(\hat{h})](s) \|_{H_0^1(\Omega)}^2 ds dt \\ &\leq \frac{C}{\alpha} \int_0^T e^{-\alpha s} \mathbb{E} \| [h - \hat{h}](s) \|_{H_0^1(\Omega)}^2 ds. \end{aligned}$$

Since the exponential weight in time provides an equivalent norm in $\mathcal{N}_w^2(0, T, H_0^1(\Omega))$, if $\alpha > C$, Ψ is a contractive mapping, it has a unique fixed-point and the result holds.

4 Annexe

4.1 Simon's theorem

Theorem 4.1. [24] Let $1 < p \leq +\infty$, $1 \leq q \leq +\infty$ and V, E and F three Banach's spaces such that $V \hookrightarrow E \hookrightarrow F$. Then, if A is a bounded subset of $W^{1,p}(0, T; F, F)$ and of $L^q(0, T; V)$, then A is relatively compact in $C([0, T]; F)$ and $L^q(0, T; E)$.

4.2 Heat equation

Lemma 4.2. Consider $T > 0$, $Q =]0, T[\times \Omega$, $u_0 \in H_0^1(\Omega)$, $G \in L^2(Q)$ and u in $W(0, T) = \{v \in L^2(0, T; H_0^1(\Omega)), \partial_t v \in L^2(0, T; H^{-1}(\Omega))\}$ the unique weak solution of the heat equation:

$$\partial_t u - \Delta u = G \text{ in } Q, \quad \text{with } u(0, \cdot) = u_0.$$

Then, $u \in H^1(Q) \cap C([0, T], H_0^1(\Omega))$ and for any $t \in [0, T]$,

$$\int_{]0, t[\times \Omega} |\partial_t u(\sigma)|^2 dx d\sigma + \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx = \frac{1}{2} \int_{\Omega} |\nabla u(0)|^2 dx + \int_{]0, t[\times \Omega} G(\sigma) \partial_t u dx d\sigma. \quad (4.1)$$

Proof. .

The result holds if the data u_0 and G are regular. This can be proved by multiplying the equation by $\partial_t u$. If u_0^n and G^n are regular approximations of u_0 and G and if u^n denotes the corresponding solution, then, for any $t \in]0, T]$, we get that

$$\begin{aligned} & \int_{]0, t[\times \Omega} |\partial_t (u^n - u^m)(\sigma)|^2 dx d\sigma + \int_{\Omega} |\nabla (u^n - u^m)(t)|^2 dx \leq \\ & \leq \int_{\Omega} |\nabla (u_0^n - u_0^m)|^2 dx + \int_{]0, t[\times \Omega} |(G^n - G^m)(\sigma)|^2 dx d\sigma, \end{aligned}$$

and one concludes by a Cauchy sequence argument.

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