A global existence and uniqueness result for a stochastic Allen-Cahn equation with constraint

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Abstract

This paper addresses the analysis of a time noise-driven Allen-Cahn equation modelling the evolution of damage in continuum media in the presence of stochastic dynamics. The nonlinear character of the equation is mainly due to a multivoque maximal monotone operator representing a constraint on the damage variable which is forced to take physically admissible values. By a Yosida approximation and a time-discretization procedure, we prove a result of global-in-time existence and uniqueness of the solution to the stochastic problem.

1 Introduction

We are interested in the following stochastic problem

$$\begin{cases} du + (\xi - \Delta u)\, dt = (w_s(u) + f)\, dt + h(u) dW & \text{in } \Omega \times D \times (0,T) \quad (1.1a) \\ u(\omega, x, t=0) = u_0(x) & \omega \in \Omega, x \in D, \quad (1.1b) \\ \nabla u.\mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0,T), \quad (1.1c) \end{cases}$$

where $\xi \in \partial I_{[0,1]}(u)$, $T > 0$, $W = \{W_t, \mathcal{F}_t, 0 \leqslant t \leqslant T\}$ is a standard adapted continuous Brownian motion defined on the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, D is a smooth bounded domain of $\mathbb{R}^d$ with $d \geqslant 1$, $\mathbf{n}$ is the outward unit normal vector to ∂D and u_0 is a given initial condition. Note that equation (1.1a) can also be written in the following way:

$$w_s(u) + f - \partial_t\Big(u - \int_0^t h(u) dW\Big) + \Delta u \in \partial I_{[0,1]}(u) \text{ in } \Omega \times D \times (0,T),$$

where the stochastic integral is understood in the sense of Itô.

Remark 1.1 *The subdifferential $\partial I_{[0,1]}$ represents a physical constraint on u which is forced to take values in the interval $[0,1]$. More precisely, we have $I_{[0,1]} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$I_{[0,1]}(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ +\infty & \text{else.} \end{cases}$$

For any $x \in [0,1]$, it results that (see e.g. [7])

$$\partial I_{[0,1]}(x) = \begin{cases} \{0\} & \text{if } x \in]0,1[, \\ \mathbb{R}^- & \text{if } x = 0, \\ \mathbb{R}^+ & \text{if } x = 1. \end{cases}$$

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We assume the following hypotheses:

H_1: $u_0 \in H^1(D)$.

H_2: $0 \leqslant u_0(x) \leqslant 1$ for almost all $x \in D$.

H_3: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function such that $h(0) = h(1) = 0$.

H_4: $w_s : \mathbb{R} \to [0, +\infty)$ is a Lipschitz-continuous function with, for convenience, $w_s(0) = 0$.

H_5: $f \in \mathcal{N}_w^2(0, T, L^2(D))^\dagger$.

1.1 General notations

For the sake of clarity, let us make precise some useful notations:

- $Q = D \times (0, T)$.
- C_h and L_s stand for the Lipschitz constants of h and w_s, respectively.
- $x.y$ the usual scalar product of x and y in $\mathbb{R}^d$.
- $V = H^1(D)$ and identify $H = L^2(D)$ with its dual space H', so that $V \hookrightarrow H \hookrightarrow V'$ with dense and compact injections. Note that (V, H, V') is a so-called *Hilbert triplet.*
- $\mathscr{D}(D) = \mathscr{C}_c^\infty(D)$ and $\mathscr{D}'(D)$ the space of distributions on D.
- $||.||$ the usual norm in $L^2(D)$.
- $\mathbb{E}[.]$ the expectation, *i.e.* the integral over Ω with respect to the probability measure $\mathbb{P}$.

1.2 The model

Equation (1.1a) is known as an Allen-Cahn type equation and it is used to describe several physical phenomena, like phase transitions. Here, we deal with the analysis of this equation, having in mind the evolution of damage in continuum media. More precisely, we assume that u represents a damage parameter, i.e. the local proportion of active cohesive bonds in the micro-structure of the material. In this direction, the function f on the right hand side of (1.1a) stands for an external source of damage (mechanical or chemical). With this interpretation, we include in the model a constraint forcing u to take values in the interval $[0, 1]$, so that $u = 1$ means that the material is completely undamaged, $u = 0$ that it is completely damaged while $u \in (0, 1)$ describes an intermediate situation. The physical constraint is ensured by the presence of a sub-differential graph, i.e. a multivoque maximal monotone operator. In addition, we include in the equation a multiplicative time noise where the noise diffusion coefficient h depends on the damage parameter itself. From a physical point of view, the presence of a random process reflects the fact that the phenomenon of damage is related to microscopic changes in the structure and configuration of the material lattice as a consequence of breaking bonds and the formation of cavities and voids. These phenomena are clearly related to stochastic processes occurring at a microscopic level (as it is introduced in Ising materials), which we aim to take into account in a macroscopic description.

The literature on the deterministic Allen-Cahn equation is very rich, also including the presence of non-smooth (monotone) operators (see, among the others, [1, 13, 14, 16, 21]). For the stochastic case, we refer to fairly recent results [2, 20], and the references therein. We underline that, in these contributions the sub-differential operator was replaced by smooth nonlinearities (possibly with a prescribed growth condition), as double-well potentials. However, we invite the reader interested in stochastic partial differential inclusions in a rather general situation to consult [5, 6, 19] and [4] concerning transition semigroup and invariant measures.

As an equation describing a phase transition process (as damage), (1.1a) may be recovered as a balance law referring to the theory introduced by Michel Frémond [17]. Such a theory relies on a generalized version of the principle of virtual powers where micro-forces

†For a given separable Hilbert space X we denote by $\mathcal{N}_w^2(0, T, X)$ the space of the predictable X-valued processes endowed with the norm $||\phi||^2_{\mathcal{N}_w^2(0,T,X)} := \mathbb{E}\left[\int_0^T ||\phi||_X^2 dt\right]$ (see Da Prato-Zabczyk [15] p.94).

and micro-motions responsible for the phase transition are included. Concerning this approach, we quote the papers [8, 9, 10, 11, 12] where volume and surface damage models are deduced and analytically investigated. According to Frémond's theory, we now detail the derivation of system (1.1)

To this aim, we introduce as state variables of the model the damage parameter u and its gradient ∇u. Thus, the free energy functional reads

$$\Psi(u, \nabla u) = \hat{w}_s(u) + \frac{1}{2}|\nabla u|^2 + I_{[0,1]}(u), \tag{1.2}$$

where the indicator function of the interval $[0,1]$ restricts the domain of the free energy to the (physically) admissible values for u. Moreover, the function $\hat{w}_s$ is related to the internal cohesion of the material and it may depend on the damage parameter itself: denoting by $w_s := -\hat{w}_s'$, we may suppose that w_s is a positive function, vanishing once $u = 0$ (the cohesion is null in the case of complete damage). Then, we introduce the dissipation of the system by a pseudo-potential of dissipation *à la Moreau*. As dissipative variable we consider the time derivative of $z = u - \int_0^t h(u)dW$. Thus, the resulting dissipation functional is

$$\Phi(z_t) = \frac{1}{2}|z_t|^2. \tag{1.3}$$

Note that, once $h = 0$, we recover exactly the choice used for deterministic damage models (rate-dependent), like e.g. in [8, 9, 12].

Neglecting acceleration effects, the generalized principle of virtual powers (in the case when no macroscopic displacements are considered) leads to the balance law

$$B - \text{div } \mathbf{H} = f, \tag{1.4}$$

supplemented by a natural boundary condition $\mathbf{H} \cdot \mathbf{n} = 0$. B and $\mathbf{H}$ are microscopic forces and stresses acting in the material specified by the following constitutive laws

$$B = \frac{\partial \Psi}{\partial u} + \frac{\partial \Phi}{\partial z_t}, \quad \mathbf{H} = \frac{\partial \Psi}{\partial \nabla u}. \tag{1.5}$$

Hence, (1.1a) and (1.1c) easily follow combining (1.2)-(1.3) with (1.4)-(1.5).

1.3 Goal of the study and outline of the paper

First of all, let us introduce the concept of solution we are interested in for Problem (1.1).

Definition 1 *Any pair (u, ξ) with u belonging to $\mathcal{N}_w^2(0,T,H^1(D)) \cap L^\infty\left(0,T,L^2\left(\Omega, H^1(D)\right)\right) \cap L^2\left(\Omega, \mathscr{C}\left(0,T;L^2(D)\right)\right)$, $0 \le u \le 1$ and ξ in $\mathcal{N}_w^2\left(0,T,L^2(D)\right)$ is a solution to the stochastic problem (1.1) if almost everywhere in $(0,T)$, $\mathbb{P}$-almost surely in Ω and for any v in V*

$$\int_D \partial_t \left(u - \int_0^\cdot h(u)dW(s)\right) v dx + \int_D \nabla u.\nabla v dx + \int_D \xi v dx = \int_D (w_s(u) + f)\, v dx,$$

with $u(.,0) = u_0$ and $\xi \in \partial I_{[0,1]}(u)$.

Remark 1 *(Sense of the initial condition) Since $u \in L^2\left(\Omega, \mathscr{C}\left(0,T;L^2(D)\right)\right)$, it satisfies the initial condition in the following sense*

$$\mathbb{P}\text{-}a.s \text{ in } \Omega,\ u(.,0) = \lim_{t\to 0} u(.,t) \text{ in } L^2(D).$$

The aim of this paper is to show the following result:

Theorem 1.2 *Under hypotheses H_1-H_5 there exists a unique pair (u, ξ) solution to Problem (1.1) in the sense of Definition 1.*

To do this, we firstly consider in Section 2 a family of approximating problems (by regularization of the maximal monotone operator $\partial I_{[0,1]}$) depending on a parameter $\epsilon > 0$ and we prove a related well-posedness result. Secondly, in Section 3, we perform the passage to the limit with respect to the approximating parameter $\epsilon > 0$, by compactness and monotonicity tools, and we show that the approximate solutions converge to a solution of Problem (1.1), in the sense of Definition 1, and that this solution is unique.

2 Regularized problem

Set $\epsilon > 0$ and consider the following stochastic problem

$$\begin{cases} du_\epsilon + (\psi_\epsilon(u_\epsilon) - \Delta u_\epsilon)\,dt = (w_s(u_\epsilon) + f)\,dt + h(u_\epsilon)dW & \text{in } \Omega \times D \times (0,T), \quad (2.1a) \\ u_\epsilon(\omega, x, t=0) = u_0(x), & \omega \in \Omega, x \in D, \quad (2.1b) \\ \nabla u_\epsilon.\mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0,T), \quad (2.1c) \end{cases}$$

where ψ_ϵ denotes the Yosida approximation of $\partial I_{[0,1]}$ (see e.g. [3, 7]). For all $v \in \mathbb{R}$ it holds

$$\psi_\epsilon(v) = -\frac{(v)^-}{\epsilon} + \frac{(v-1)^+}{\epsilon} = \begin{cases} \dfrac{v}{\epsilon} & \text{if} \quad v \leqslant 0 \\ 0 & \text{if} \quad v \in [0,1] \\ \dfrac{v-1}{\epsilon} & \text{if} \quad v \geqslant 1. \end{cases}$$

Definition 2 *A function $u_\epsilon \in L^2\left(\Omega, \mathscr{C}\left(0,T; L^2(D)\right)\right) \cap L^\infty\left(0,T; L^2(\Omega, H^1(D))\right) \cap \mathcal{N}^2_w\big(0,T,H^1(D)\big)$ such that $\partial_t\big(u_\epsilon - \int_0^\cdot h(u_\epsilon)dW\big)$ and Δu_ϵ belong to $L^2(\Omega \times Q)$, is a solution to the stochastic problem (2.1) if almost everywhere in $(0,T)$ and $\mathbb{P}$-almost surely in Ω, the following variational formulation holds for any $v \in H^1(D)$*

$$\int_D \partial_t\left(u_\epsilon - \int_0^\cdot h(u_\epsilon)dW(s)\right)vdx + \int_D \nabla u_\epsilon.\nabla v dx + \int_D \psi_\epsilon(u_\epsilon)vdx = \int_D \big(w_s(u_\epsilon) + f\big)vdx,$$

with $u_\epsilon(.,0) = u_0$.

Remark 2 *Since $u_\epsilon \in L^2\left(\Omega, \mathscr{C}\left(0,T; L^2(D)\right)\right)$, it satisfies the initial condition in the sense of Remark 1.*

We have the following well-posedness result:

Theorem 2.1 *Under hypotheses H_1-H_5 and for any $\epsilon > 0$, there exists a unique solution u_ϵ in the sense of Definition 2 to the stochastic Problem (2.1).*

2.1 Existence of u_ϵ

Note that the well-posedness of Problem (2.1) could be deduced from classic results in the literature ([15, 18]), but for a matter of self-containedness and to prepare the *a priori* estimates needed to pass to the limit on the penalization procedure, we propose to detail a result of existence of a solution based on an implicit time discretization scheme for the deterministic part and an explicit one for the Itô part.

We consider a positive integer N and denote by $\Delta t = \frac{T}{N}$ and $t_n = n\Delta t$, $n \in \{0, ..., N\}$. Let us first introduce some classical notations needed in the sequel.

Definition 3 *For any sequence $(x_n) \subset X$, where X is a separable Banach space, let us denote by*

$$\begin{aligned} x^{\Delta t} &= \sum_{k=0}^{N-1} x_{k+1}\mathbb{1}_{[t_k, t_{k+1})}, \\ \tilde{x}^{\Delta t} &= \sum_{k=0}^{N-1} \left[\frac{x_{k+1} - x_k}{\Delta t}(t - t_k) + x_k\right]\mathbb{1}_{[t_k, t_{k+1})}, \\ \frac{\partial \tilde{x}^{\Delta t}}{\partial t} &= \sum_{k=0}^{N-1} \frac{x_{k+1} - x_k}{\Delta t}\mathbb{1}_{[t_k, t_{k+1})}, \end{aligned}$$

and elementary calculations yield for an arbitrary constant $C > 0$ independent of Δt

$$\|x^{\Delta t}\|^2_{L^2(0,T;X)} = \Delta t \sum_{k=1}^{N} \|x_k\|^2_X \quad ; \quad \|\tilde{x}^{\Delta t}\|^2_{L^2(0,T;X)} \leqslant C\Delta t \sum_{k=0}^{N} \|x_k\|^2_X;$$
$$\|x^{\Delta t} - \tilde{x}^{\Delta t}\|^2_{L^2(0,T;X)} = \Delta t \sum_{k=0}^{N-1} \|x_{k+1} - x_k\|^2_X;$$
$$\|\frac{\partial \tilde{x}^{\Delta t}}{\partial t}\|^2_{L^2(0,T;X)} = \frac{1}{\Delta t} \sum_{k=0}^{N-1} \|x_{k+1} - x_k\|^2_X;$$
$$\|x^{\Delta t}\|_{L^\infty(0,T;X)} = \max_{k=1,..,N} \|x_k\|_X \quad ; \quad \|\tilde{x}^{\Delta t}\|_{L^\infty(0,T;X)} = \max_{k=0,..,N} \|x_k\|_X.$$

The discretization scheme is the following one: for given small positive parameter Δt and u_n in $L^2\left((\Omega, \mathcal{F}_{t_n}); H^1(D)\right)$ $(n \geq 0)$, we aim to find u_{n+1} in $L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)\right)$, such that $\mathbb{P}$-a.s in Ω and for any v in $H^1(D)$

$$\begin{aligned}
&\int_D (u_{n+1} - u_n) v dx + \Delta t \int_D \big(\nabla u_{n+1}.\nabla v + \psi_\epsilon(u_{n+1}) v\big) dx \\
= \quad &\Delta t \int_D (w_s(u_{n+1}) + f_n) v dx + (W_{n+1} - W_n) \int_D h(u_n) v dx, \qquad (2.2)
\end{aligned}$$

where $W_n = W(t_n)$ and $f_n = \dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n} f(s) ds$[†]. To proceed in this way we prove that, once n is fixed, we can find the solution for the step $n+1$ by a fixed point argument, once it is ensured that the time step of the discrete scheme is sufficiently small.

Lemma 1 *Set $u_n \in L^2\left((\Omega, \mathcal{F}_{t_n}); H^1(D)\right)$. If $\Delta t < \dfrac{1}{\left(L_s + \frac{1}{\epsilon}\right)}$, there exists a unique $u_{n+1} \in L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)\right)$, satisfying the variational problem (2.2).*

Proof. Consider $\mathbb{T}$ the application defined by

$$\begin{aligned}
\mathbb{T} : L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D)\right) &\to L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)\right) \\
S &\mapsto \mathbb{T}(S),
\end{aligned}$$

where for any $S \in L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D)\right)$, $u = \mathbb{T}(S)$ is the solution in $L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)\right)$ of the variational problem

$$\begin{aligned}
&\mathbb{E}\left[\int_D (u - u_n) v dx\right] + \Delta t \mathbb{E}\left[\int_D \nabla u.\nabla v + \psi_\epsilon(S) v dx\right] \\
= \quad &\Delta t \mathbb{E}\left[\int_D (w_s(S) + f_n) v dx\right] + \mathbb{E}\left[(W_{n+1} - W_n) \int_D h(u_n) v dx\right], \ \forall v \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)).
\end{aligned}$$

By denoting $B = u_n + \Delta t\big(w_s(S) + f_n - \psi_\epsilon(S)\big) + (W_{n+1} - W_n) h(u_n) \in L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D)\right)$, this variational problem can be rewritten as

$$\mathbb{E}\left[\int_D uv + \Delta t \nabla u.\nabla v dx\right] = \mathbb{E}\left[\int_D B v dx\right], \ \forall v \in L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)\right).$$

Thanks to the theorem of Lax-Milgram, $\mathbb{T}$ is a well-defined function. Moreover, for any $S_1, S_2 \in L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D)\right)$, by denoting $u_1 = \mathbb{T}(S_1)$ and $u_2 = \mathbb{T}(S_2)$, one has that for any v in $L^2\left((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)\right)$

$$\mathbb{E}\left[\int_D (u_1 - u_2) v + \Delta t \nabla(u_1 - u_2).\nabla v dx\right] = \mathbb{E}\left[\int_D (B_1 - B_2) v dx\right],$$

where

$$B_1 = u_n + \Delta t\big(w_s(S_1) + f_n - \psi_\epsilon(S_1)\big) + (W_{n+1} - W_n) h(u_n),$$

[†]For convenience, one denotes $t_{-1} = -\Delta t$ and f is extended by 0 for negative t.

and

$$B_2 = u_n + \Delta t\big(w_s(S_2) + f_n - \psi_\epsilon(S_2)\big) + (W_{n+1} - W_n)h(u_n).$$

Then, for $v = u_1 - u_2$, we have

$$\begin{aligned}
\mathbb{E}\Big[\int_D |u_1-u_2|^2dx\Big] &\leq \mathbb{E}\Big[\int_D |u_1-u_2|^2 + \Delta t|\nabla(u_1-u_2)|^2dx\Big]\\
&= \Delta t\mathbb{E}\Big[\int_D \big(w_s(S_1) - w_s(S_2) + \psi_\epsilon(S_2) - \psi_\epsilon(S_1)\big)(u_1-u_2)dx\Big]\\
&\leqslant \Delta t\Big(L_s + \frac{1}{\epsilon}\Big)\mathbb{E}\Big[\int_D |u_1-u_2||S_1-S_2|dx\Big],
\end{aligned}$$

which yields

$$\|u_1-u_2\|_{L^2((\Omega,\mathcal{F}_{t_{n+1}});L^2(D))} \leqslant \Delta t\Big(L_s + \frac{1}{\epsilon}\Big)\|S_1-S_2\|_{L^2((\Omega,\mathcal{F}_{t_{n+1}});L^2(D))}.$$

Then, for $\Delta t < \dfrac{1}{(L_s + \frac{1}{\epsilon})}$, $\mathbb{T}$ is a contractive mapping in $L^2\big((\Omega,\mathcal{F}_{t_{n+1}});L^2(D)\big)$ and the result holds. ■

Proposition 2.2 *(Estimates on the approximate sequences)*
There exists a constant $C>0$ which only depends on T, C_h, u_0, L_s and f such that

$$\begin{aligned}
\|u^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))},\ \|\tilde{u}^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))} &\leqslant C\\
\|\nabla u^{\Delta t}\|_{L^2(\Omega\times Q)} &\leqslant C\\
\|u^{\Delta t} - \tilde{u}^{\Delta t}\|_{L^2(\Omega\times Q)} &\leqslant C\Delta t.
\end{aligned}$$

Proof. By using the test function $v = u_{n+1}$ in (2.2), we have

$$\begin{aligned}
&\int_D (u_{n+1}-u_n)u_{n+1}dx + \Delta t\int_D |\nabla u_{n+1}|^2 + \psi_\epsilon(u_{n+1})u_{n+1}dx\\
=\ &\Delta t\int_D (w_s(u_{n+1}) + f_n)\,u_{n+1}dx + (W_{n+1}-W_n)\int_D h(u_n)u_{n+1}dx.
\end{aligned}$$

Using the formula $ab = \frac{1}{2}\left[a^2 + b^2 - (a-b)^2\right]$ with $a = u_{n+1} - u_n$ and $b = u_{n+1}$ yields

$$\begin{aligned}
&\frac{1}{2}\int_D |u_{n+1}-u_n|^2dx + \frac{1}{2}\int_D |u_{n+1}|^2dx - \frac{1}{2}\int_D |u_n|^2dx + \Delta t\int_D |\nabla u_{n+1}|^2 + \psi_\epsilon(u_{n+1})u_{n+1}dx\\
=\ &\Delta t\int_D (w_s(u_{n+1}) + f_n)\,u_{n+1}dx + (W_{n+1}-W_n)\int_D h(u_n)u_ndx\\
&+ (W_{n+1}-W_n)\int_D h(u_n)(u_{n+1}-u_n)dx.
\end{aligned}$$

Thanks to the monotonicity of ψ_ϵ and the fact that $\psi_\epsilon(0) = 0$, we get for any $\delta > 0$

$$\begin{aligned}
&\frac{1}{2}\int_D |u_{n+1}-u_n|^2dx + \frac{1}{2}\int_D |u_{n+1}|^2dx - \frac{1}{2}\int_D |u_n|^2dx + \Delta t\int_D |\nabla u_{n+1}|^2dx\\
\leqslant\ &\Delta t\int_D (w_s(u_{n+1}) + f_n)\,(u_{n+1} - u_n + u_n)\,dx + (W_{n+1}-W_n)\int_D h(u_n)u_ndx\\
&+\frac{1}{2\delta}(W_{n+1}-W_n)^2\int_D h^2(u_n)dx + \frac{\delta}{2}\int_D |u_{n+1}-u_n|^2dx.
\end{aligned}$$

Then by taking the expectation, we get since $\mathbb{E}\Big[(W_{n+1}-W_n)\int_D h(u_n)u_ndx\Big] = 0$

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\int_D |u_{n+1}-u_n|^2 + |u_{n+1}|^2 - |u_n|^2dx\Big] + \Delta t\mathbb{E}\Big[\int_D |\nabla u_{n+1}|^2dx\Big]\\
\leqslant\ &\frac{\Delta t}{2}\mathbb{E}\Big[\int_D L_s^2|u_{n+1}|^2 + |u_{n+1}-u_n|^2 + L_s^2|u_{n+1}|^2 + |u_n|^2dx\Big]\\
&+\frac{\Delta t}{2}\mathbb{E}\Big[\int_D |f_n|^2 + |u_{n+1}-u_n|^2 + |f_n|^2 + |u_n|^2dx\Big]\\
&+\frac{1}{2\delta}\mathbb{E}\big[(W_{n+1}-W_n)^2\big]\mathbb{E}\Big[\int_D h^2(u_n)dx\Big] + \frac{\delta}{2}\mathbb{E}\Big[\int_D |u_{n+1}-u_n|^2dx\Big].
\end{aligned}$$

And, e.g. for $\delta = \frac{1}{2}$

$$\frac{1}{4}\mathbb{E}\Big[\int_D |u_{n+1}-u_n|^2dx\Big] + \big(\frac{1}{2} - \Delta t L_s^2\big)\mathbb{E}\Big[\int_D |u_{n+1}|^2 - |u_n|^2 dx\Big] + \Delta t\mathbb{E}\Big[\int_D |\nabla u_{n+1}|^2 dx\Big]$$
$$\leqslant \Delta t\mathbb{E}\Big[\int_D |u_{n+1}-u_n|^2dx\Big] + \Delta t\big(1+L_s^2\big)\mathbb{E}\Big[\int_D |u_n|^2dx\Big] + \Delta t\mathbb{E}\Big[\int_D |f_n|^2dx\Big] + \Delta t\mathbb{E}\Big[\int_D h^2(u_n)dx\Big],$$

where we used the fact that $\mathbb{E}\big[(W_{n+1}-W_n)^2\big] = \Delta t$.
Thus, for Δt small enough, we can infer that

$$\begin{aligned}&\frac{1}{8}\mathbb{E}\Big[\int_D |u_{n+1}-u_n|^2dx\Big] + \frac{1}{4}\mathbb{E}\Big[\int_D |u_{n+1}|^2 - |u_n|^2 dx\Big] + \Delta t\mathbb{E}\Big[\int_D |\nabla u_{n+1}|^2 dx\Big]\\ \leqslant\ & \Delta t\big(1+C_h^2+L_s^2\big)\mathbb{E}\Big[\int_D |u_n|^2dx\Big] + \Delta t\mathbb{E}\Big[\int_D |f_n|^2dx\Big].\end{aligned}$$

Consequently, for any $k \in \{0,..,N\}$ if one denotes by $\|.\|$ the norm in $L^2(D)$

$$\begin{aligned}&\frac{1}{8}\sum_{n=0}^{k-1}\mathbb{E}\left[\|u_{n+1}-u_n\|^2\right] + \frac{1}{4}\mathbb{E}\left[\|u_k\|^2\right] + \Delta t\sum_{n=0}^{k-1}\mathbb{E}\left[\|\nabla u_{n+1}\|^2\right]\\ \leqslant\ & \frac{1}{4}\mathbb{E}\left[\|u_0\|^2\right] + \Delta t\big(1+C_h^2+L_s^2\big)\sum_{n=0}^{k-1}\mathbb{E}\left[\|u_n\|^2\right] + \Delta t\sum_{n=0}^{k-1}\mathbb{E}\left[\|f_n\|^2\right].\end{aligned}$$

Note that

$$\begin{aligned}\Delta t\sum_{n=0}^{k-1}\mathbb{E}\left[\|f_n\|^2\right] &= \Delta t\sum_{n=0}^{k-1}\mathbb{E}\Big[\|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} f(s)ds\|^2\Big]\\ &= \frac{1}{\Delta t}\sum_{n=0}^{k-1}\mathbb{E}\Big[\int_D\Big(\int_{t_{n-1}}^{t_n} f(s)ds\Big)^2dx\Big]\\ &\leqslant \frac{\Delta t}{\Delta t}\sum_{n=0}^{k-1}\mathbb{E}\Big[\int_D\int_{t_{n-1}}^{t_n}\big|f(s)\big|^2dsdx\Big]\\ &\leqslant \|f\|^2_{L^2(\Omega\times Q)}.\end{aligned}$$

The discrete Gronwall lemma asserts then that by denoting $\tilde{C} = 4\|f\|^2_{L^2(\Omega\times Q)} + \|u_0\|^2_{L^2(D)}$

$$\mathbb{E}\left[\|u_k\|^2\right] \leqslant \tilde{C}\exp\Big(\sum_{n=0}^{k-1}4\Delta t\left(1+C_h^2+L_s^2\right)\Big) \leqslant \tilde{C}e^{4T\left(1+C_h^2+L_s^2\right)}.$$

In this way,

$$\begin{aligned}&\frac{1}{4}\mathbb{E}\left[\|u_k\|^2\right] + \frac{1}{8}\sum_{n=0}^{k-1}\mathbb{E}\left[\|u_{n+1}-u_n\|^2\right] + \Delta t\sum_{n=0}^{k-1}\mathbb{E}\left[\|\nabla u_{n+1}\|^2\right]\\ \leqslant\ & \frac{1}{4}\|u_0\|^2 + \|f\|^2_{L^2(\Omega\times Q)} + \tilde{C}T\left(1+C_h^2+L_s^2\right)e^{4T\left(1+C_h^2+L_s^2\right)}.\end{aligned}$$

Using the notations of Definition 3, we conclude that there exists a constant $C > 0$ only depending on T, C_h, u_0, L_s and f such that

$$\begin{aligned}\|u^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))}, \|\tilde{u}^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))} &\leqslant C\\ \|\nabla u^{\Delta t}\|_{L^2(\Omega\times Q)} &\leqslant C\\ \|u^{\Delta t} - \tilde{u}^{\Delta t}\|_{L^2(\Omega\times Q)} &\leqslant C\sqrt{\Delta t},\end{aligned}$$

which concludes the proof. ∎

Proposition 2.3 *(Additional estimates)*
There exists a constant $C > 0$ which only depends on T, C_h, u_0, L_s and f and such that

$$\begin{aligned}\|u^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega;H^1(D)))}, \|\tilde{u}^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega;H^1(D)))} &\leqslant C(1+\|\psi_\epsilon(u^{\Delta t})\|_{L^2(\Omega\times Q)}),\\ \|\tilde{B}^{\Delta t}\|_{L^2(\Omega\times(0,T);H^1(D))}, \ \|\partial_t(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega\times Q)} &\leqslant C(1+\|\psi_\epsilon(u^{\Delta t})\|_{L^2(\Omega\times Q)}),\\ \|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{L^2(0,T;L^2(\Omega;H^1(D)))} &\leqslant C(1+\|\psi_\epsilon(u^{\Delta t})\|_{L^2(\Omega\times Q)})\sqrt{\Delta t},\end{aligned}$$

where $\tilde{B}^{\Delta t}$ *is given by Definition 3 with* $B_n = \int_0^{t_n} h(u^{\Delta t}(s-\Delta t))dW(s)$, $n \in \{0,..,N\}$.

Proof. By denoting as previously the norm in $L^2(D)$ by $||.||$, setting the test function $v = u_{n+1} - u_n - (W_{n+1} - W_n)h(u_n)$ in (2.2) yields for any $\delta > 0$, after taking the expectation

$$\begin{aligned}
&\mathbb{E}\left[||v||^2\right] - \delta\frac{\Delta t^2}{2}\mathbb{E}\left[||\psi_\epsilon(u_{n+1})||^2\right] - \frac{1}{2\delta}\mathbb{E}\left[||v||^2\right] \\
&+\frac{\Delta t}{2}\mathbb{E}\Big[||\nabla u_{n+1}||^2 + \frac{1}{2}||\nabla u_{n+1} - \nabla u_n||^2 - ||\nabla u_n||^2 - 2\Delta t||\nabla h(u_n)||^2\Big] \\
\leqslant\ & \mathbb{E}\left[||v||^2\right] + \Delta t\mathbb{E}\Big[\int_D \psi_\epsilon(u_{n+1})v dx\Big] + \Delta t\mathbb{E}\Big[\int_D \nabla u_{n+1}.\nabla v dx\Big] \\
=\ & \Delta t\mathbb{E}\Big[\int_D (w_s(u_{n+1}) + f_n)\, v dx\Big] \\
\leqslant\ & \delta\frac{\Delta t^2}{2}\left(L_s^2\mathbb{E}\left[||u_{n+1}||^2\right] + \mathbb{E}\left[||f_n||^2\right]\right) + \frac{1}{\delta}\mathbb{E}\left[||v||^2\right].
\end{aligned}$$

Indeed, since $\mathbb{E}\Big[(W_{n+1} - W_n)\int_D \nabla u_n.\nabla h(u_n)dx\Big] = 0$ and $\mathbb{E}\left[(W_{n+1} - W_n)^2\right] = \Delta t$, we have

$$\begin{aligned}
\mathbb{E}\Big[\int_D \nabla u_{n+1}.\nabla v dx\Big] =\ & \frac{1}{2}\Big(\mathbb{E}\left[||\nabla u_{n+1}||^2\right] + \mathbb{E}\left[||\nabla(u_{n+1} - u_n)||^2\right] - \mathbb{E}\left[||\nabla u_n||^2\right]\Big) \\
& -\mathbb{E}\Big[(W_{n+1} - W_n)\int_D \nabla(u_{n+1} - u_n).\nabla h(u_n)dx\Big] \\
\geqslant\ & \frac{1}{2}\Big(\mathbb{E}\left[||\nabla u_{n+1}||^2\right] + \frac{1}{2}\mathbb{E}\left[||\nabla(u_{n+1} - u_n)||^2\right] - \mathbb{E}\left[||\nabla u_n||^2\right] - 2\Delta t\mathbb{E}\left[||\nabla h(u_n)||^2\right]\Big).
\end{aligned}$$

Finally, for $\delta = 3$, one gets

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[||v||^2\right] + \frac{\Delta t}{2}\mathbb{E}\Big[||\nabla u_{n+1}||^2 + \frac{1}{2}||\nabla u_{n+1} - \nabla u_n||^2 - ||\nabla u_n||^2\Big] \\
\leqslant\ & 3\frac{\Delta t^2}{2}\left(\mathbb{E}\left[||\psi_\epsilon(u_{n+1})||^2\right] + L_s^2\mathbb{E}\left[||u_{n+1}||^2\right]\right) + 3\frac{\Delta t^2}{2}\mathbb{E}\left[||f_n||^2\right] + \Delta t^2 C_h^2\mathbb{E}\left[||\nabla u_n||^2\right].
\end{aligned}$$

Then, for any $k \in \{0, ..., N\}$

$$\begin{aligned}
&\sum_{n=0}^{k-1}\mathbb{E}\left[||v||^2\right] + \Delta t\mathbb{E}\left[||\nabla u_k||^2\right] + \frac{\Delta t}{2}\sum_{n=0}^{k-1}\mathbb{E}\left[||\nabla u_{n+1} - \nabla u_n||^2\right] \\
\leqslant\ & \Delta t||\nabla u_0||^2 + 3\Delta t^2\left(\sum_{n=0}^{k-1}\mathbb{E}\left[||\psi_\epsilon(u_{n+1})||^2\right] + L_s^2\mathbb{E}\left[||u_{n+1}||^2\right]\right) + 3\Delta t^2\sum_{n=0}^{k-1}\mathbb{E}\left[||f_n||^2\right] + 2\Delta t^2 C_h^2\sum_{n=0}^{k-1}\mathbb{E}\left[||\nabla u_n||^2\right],
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{n=0}^{k-1}\Delta t\mathbb{E}\Big[\Big|\Big|\frac{v}{\Delta t}\Big|\Big|^2\Big] + \mathbb{E}\left[||\nabla u_k||^2\right] + \frac{1}{2}\sum_{n=0}^{k-1}\mathbb{E}\left[||\nabla u_{n+1} - \nabla u_n||^2\right] \\
\leqslant\ & ||\nabla u_0||^2 + 3\Delta t\left(\sum_{n=0}^{k-1}\mathbb{E}\left[||\psi_\epsilon(u_{n+1})||^2\right] + L_s^2\mathbb{E}\left[||u_{n+1}||^2\right]\right) + 3\Delta t\sum_{n=0}^{k-1}\mathbb{E}\left[||f_n||^2\right] + 2\Delta t C_h^2\sum_{n=0}^{k-1}\mathbb{E}\left[||\nabla u_n||^2\right] \\
\leqslant\ & \left(1 + 2C_h^2\right)||\nabla u_0||^2 + 3\left(||\psi_\epsilon(u^{\Delta t})||^2_{L^2(\Omega\times Q)} + L_s^2||u^{\Delta t}||^2_{L^2(\Omega\times Q)}\right) + 3||f||^2_{L^2(\Omega\times Q)} + 2C_h^2||\nabla u^{\Delta t}||^2_{L^2(\Omega\times Q)}.
\end{aligned}$$

Using estimates given by Proposition 2.2, we deduce by denoting

$$\begin{aligned}
\tilde{B}^{\Delta t} &= \sum_{k=0}^{N-1}\left(\frac{B_{k+1} - B_k}{\Delta t}(. - t_k) + B_k\right)\mathbb{1}_{[t_k,t_{k+1})}, \\
\text{with } B_k &= \int_0^{t_k} h(u^{\Delta t}(s-\Delta t))dW(s) = \sum_{n=0}^{k-1}(W_{n+1} - W_n)\, h(u_n),\ k \in \{0,..,N\},
\end{aligned}$$

that $u^{\Delta t}$ and $\tilde{u}^{\Delta t}$ are bounded in $L^\infty\left(0,T;L^2(\Omega;H^1(D))\right)$, that $\partial_t(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})$ is bounded in $L^2(\Omega\times Q)$ and that $\tilde{u}^{\Delta t} - u^{\Delta t} \to 0$ in $L^2\left(0,T;L^2(\Omega;H^1(D))\right)$ as $\Delta t \to 0$.

Since, for any $k \in \{0,...,N-1\}$, $E\Big[\| \sum_{n=0}^{k-1} (W_{n+1} - W_n)\, h(u_n)\|^2\Big] = \Delta t \sum_{n=0}^{k-1} E\left[\|h(u_n)\|^2\right]$, by Itô's isometry ([18] Proposition 2.3.5 p.25), one gets

$$\begin{aligned}
E\Big[\int_0^T \|\tilde{B}^{\Delta t}(t)\|^2 dt\Big] &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} E\left[\left\|\frac{B_{k+1} - B_k}{\Delta t}(t - t_k) + B_k\right\|^2\right] dt \\
&\leq C \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} E\left[\|B_{k+1} - B_k\|^2\right] + E\left[\|B_k\|^2\right] dt \\
&= C \sum_{k=0}^{N-1} \Delta t \Big\{ E\left[\|\,(W_{k+1} - W_k)\, h(u_k)\|^2\right] + E\Big[\| \sum_{n=0}^{k-1} (W_{n+1} - W_n)\, h(u_n)\|^2\Big]\Big\} \\
&\leq C \sum_{k=0}^{N-1} \Delta t^2 \sum_{n=0}^{k} E\Big[\|h(u_n)\|^2\Big].
\end{aligned}$$

Thus,

$$E\Big[\int_0^T \|\tilde{B}^{\Delta t}(t)\|^2 dt\Big] \leq C \Delta t \sum_{n=0}^{N-1} E\left[\|h(u_n)\|^2\right]$$

is bounded and the proposition is proved since the same argument holds with the H^1 norm.
■

Proposition 2.4 *There exists* $u_\epsilon \in L^2\left(\Omega, \mathscr{C}\left(0,T;L^2(D)\right)\right) \cap L^\infty\left(0,T;L^2(\Omega,H^1(D))\right) \cap \mathcal{N}_w^2\big(0,T,H^1(D)\big)$, h_ϵ *and* χ_ϵ *in* $L^2\left(0,T;L^2(\Omega,H^1(D))\right)$ *and* w_ϵ *in* $L^2(\Omega \times Q)$ *such that, up to subsequences denoted in the same way, as* $\Delta t \to 0$ *(i.e.* $N \to +\infty$*)*

$$\begin{aligned}
u^{\Delta t}, \tilde{u}^{\Delta t} &\overset{*}{\rightharpoonup} u_\epsilon \text{ in } L^\infty\left(0,T;L^2(\Omega,H^1(D))\right), \\
h(u^{\Delta t}) &\rightharpoonup h_\epsilon \text{ in } L^2\left(0,T;L^2(\Omega,H^1(D))\right), \\
\psi_\epsilon(u^{\Delta t}) &\rightharpoonup \chi_\epsilon \text{ in } L^2\left(0,T;L^2(\Omega,H^1(D))\right), \\
w_s(u^{\Delta t}) &\rightharpoonup w_\epsilon \text{ in } L^2(\Omega \times Q), \\
\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t} &\rightharpoonup u_\epsilon - \int_0^{\cdot} h_\epsilon(s) dW(s) \text{ in } L^2\left(\Omega, H^1(Q)\right), \\
\forall t \in [0,T],\ \tilde{u}^{\Delta t}(t) &\rightharpoonup u_\epsilon(t) \text{ in } L^2\left(\Omega \times D\right).
\end{aligned}$$

Proof. Note that thanks to the estimates given by Proposition 2.2 and Proposition 2.3, and noticing that $\|\psi_\epsilon(u^{\Delta t})\|_{L^2(\Omega\times Q)}$ is bounded with respect to the parameter Δt, by compactness, the existence of the weak limits $u_\epsilon, h_\epsilon, \chi_\epsilon$ and w_ϵ is immediate. We mention that u_ϵ is a $H^1(D)$-valued predictable process as weak limit of the continuous and adapted process $\tilde{u}^{\Delta t}(. - \Delta t)$ belonging to the Hilbert space $\mathcal{N}_w^2\big(0,T,H^1(D)\big)$ equipped with the norm of $L^2\left((0,T)\times\Omega, H^1(D)\right)$.
Let us now focus on the weak convergence of $\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}$ towards $u_\epsilon - \int_0^{\cdot} h_\epsilon(s) dW(s)$ in $L^2\left(\Omega, H^1(Q)\right)$.
Firstly, since $\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}, \nabla(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})$ and $\partial_t(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})$ are bounded in $L^2(\Omega \times Q)$, there exists ζ_ϵ in $L^2\left(\Omega, H^1(Q)\right)$ such that, up to a subsequence,

$$\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t} \rightharpoonup \zeta_\epsilon \text{ in } L^2\left(\Omega, H^1(Q)\right).$$

Secondly, we show below that $\tilde{B}^{\Delta t}$ converges weakly to $\int_0^{\cdot} h_\epsilon(s) dW(s)$ in $\mathscr{C}\left(0,T;L^2\left(\Omega \times D\right)\right)$.
In this way :

$$\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t} \rightharpoonup u_\epsilon - \int_0^{\cdot} h_\epsilon(s) dW(s) \text{ in } L^2\left(0,T;L^2\left(\Omega \times D\right)\right),$$

and then $\zeta_\epsilon = u_\epsilon - \int_0^\cdot h_\epsilon(s)dW(s)$. Indeed, for any $t \in [t_n, t_{n+1})$, we have

$$\begin{aligned}
& \mathbb{E}\left[\left\|\tilde{B}^{\Delta t}(t) - \int_0^t h(u^{\Delta t}(s-\Delta t))dW(s)\right\|^2\right] \\
= & \mathbb{E}\left[\left\|\frac{t-t_n}{\Delta t}\int_{t_n}^{t_{n+1}} h(u^{\Delta t}(s-\Delta t))dW(s) - \int_{t_n}^t h(u^{\Delta t}(s-\Delta t))dW(s)\right\|^2\right] \\
= & \mathbb{E}\left[\left\|(W_{n+1}-W_n)\,h(u_n)\frac{t-t_n}{\Delta t} - (W(t)-W_n)\,h(u_n)\right\|^2\right] \\
= & \mathbb{E}\left[\|h(u_n)\|^2\right]\times\left\{\mathbb{E}\left[(W_{n+1}-W_n)^2\right]\left(\frac{t-t_n}{\Delta t}\right)^2 + \mathbb{E}\left[(W(t)-W_n)^2\right]\right\} \\
& -2\mathbb{E}\left[\|h(u_n)\|^2(W_{n+1}-W_n)\frac{t-t_n}{\Delta t}(W(t)-W_n)\right] \\
= & \mathbb{E}\left[\|h(u_n)\|^2\right]\times\left\{\frac{(t-t_n)^2}{\Delta t} + (t-t_n) - 2\frac{(t-t_n)2}{\Delta t}\right\} \\
& -2\mathbb{E}\left[\|h(u_n)\|^2(W_{n+1}-W(t))\frac{t-t_n}{\Delta t}(W(t)-W_n)\right] \\
= & \mathbb{E}\left[\|h(u_n)\|^2\right]\times\left\{(t-t_n) - \frac{(t-t_n)^2}{\Delta t}\right\} \\
\leqslant & C\Delta t,
\end{aligned}$$

by using again the fact that $\mathbb{E}\left[(W_{n+1}-W_n)^2\right] = \Delta t$, $\mathbb{E}\left[(W(t)-W_n)^2\right] = t - t_n$, and the independence between the increment $W_{n+1} - W(t)$ and any $\mathcal{F}_t$- measurable process ([15] p.90).

Moreover, $h(u^{\Delta t}(.-\Delta t))$ converges weakly to h_ϵ in $L^2(\Omega\times Q)$ and since the stochastic integral

$$\begin{aligned}
I : L^2(\Omega\times Q) & \to \mathscr{C}\left(0,T;L^2(\Omega,L^2(D))\right) \\
v & \mapsto I(v):(\omega,x,t)\mapsto\int_0^t v(\omega,x,s)dW(s),
\end{aligned}$$

is linear and continuous, we thus have

$$\int_0^\cdot h(u^{\Delta t}(s-\Delta t))dW(s) \rightharpoonup \int_0^\cdot h_\epsilon(s)dW(s) \text{ in } \mathscr{C}\left(0,T;L^2(\Omega,L^2(D))\right),$$

and so

$$\tilde{B}^{\Delta t} \rightharpoonup \int_0^\cdot h_\epsilon(s)dW(s) \text{ in } \mathscr{C}\left(0,T;L^2(\Omega,L^2(D))\right).$$

Finally, since $\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}$ converges weakly in $L^2\left(\Omega, H^1(Q)\right)$[§], it converges weakly in

$$L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)\hookrightarrow\mathscr{C}\left(0,T;L^2(\Omega,L^2(D))\right),$$

then for any $t\in[0,T]$, $\tilde{u}^{\Delta t}(t)\rightharpoonup u_\epsilon(t)$ in $L^2(\Omega\times D)$. Now, by the property of the stochastic integral ([15] Theorem 4.12 p.101), $\int_0^\cdot h_\epsilon(s)dW(s)\in L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)$, which allows us to conclude that $u_\epsilon\in L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)$. ■

Remark 3 *Note that for $t = 0$ we have $\tilde{u}^{\Delta t}(0) = u_0 \rightharpoonup u_\epsilon(0)$ in $L^2(\Omega\times D)$ and then $u_0 = u_\epsilon(0)$ (in the sense of Remark 1).*

By denoting $f_{\Delta t} = \sum_{k=0}^{N-1} f_k \mathbb{1}_{[t_k,t_{k+1}[}$, we have for any v in $H^1(D)$ and $\mathbb{P}$ a-s. in Ω

$$\int_D \partial_t(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})vdx + \int_D \nabla u^{\Delta t}.\nabla vdx + \int_D \psi_\epsilon(u^{\Delta t})vdx = \int_D \left(w_s(u^{\Delta t}) + f_{\Delta t}\right)vdx.$$

[§] we remind that $H^1(Q)\hookrightarrow\mathscr{C}([0,T],L^2(D))$

Since $f_{\Delta t} \to f$ in $L^2(\Omega \times Q)$, at the limit one gets that, for any v in $H^1(D)$, α in $L^2(0,T)$ and any β in $L^2(\Omega)$, the following variational formulation holds

$$\int_{\Omega\times Q} \partial_t\left(u_\epsilon - \int_0^{\cdot} h_\epsilon(s)dW(s)\right) v\alpha\beta dxdtd\mathbb{P} + \int_{\Omega\times Q} (\nabla u_\epsilon.\nabla v)\,\alpha\beta dxdtd\mathbb{P}$$
$$+ \int_{\Omega\times Q} \chi_\epsilon v\alpha\beta dxdtd\mathbb{P} = \int_{\Omega\times Q} (w_\epsilon + f)\,v\alpha\beta dxdtd\mathbb{P}. \tag{2.3}$$

Since $H^1(D)$ is separable, one gets that almost everywhere in $(0,T)$, $\mathbb{P}$-almost surely in Ω and for any v in $H^1(D)$

$$\int_D \partial_t\left(u_\epsilon - \int_0^{\cdot} h_\epsilon(s)dW(s)\right) vdx + \int_D \nabla u_\epsilon.\nabla vdx + \int_D \chi_\epsilon vdx = \int_D (w_\epsilon + f)\,vdx.$$

Particularly, almost everywhere in $(0,T)$, $\mathbb{P}$-almost surely in Ω, one gets that

$$\Delta u_\epsilon = \partial_t\left(u_\epsilon - \int_0^{\cdot} h_\epsilon(s)dW(s)\right) + \chi_\epsilon - w_\epsilon - f,$$

a priori in V', but then in $L^2(D)$ since $L^2(D)$ is the chosen pivot-space. Thus $\Delta u_\epsilon \in L^2(\Omega\times Q)$ and u_ϵ is an Itô process of the following form

$$du_\epsilon + (\chi_\epsilon - \Delta u_\epsilon)dt = (w_\epsilon + f)\,dt + h_\epsilon dW.$$

Proposition 2.5 *The following convergences hold strongly in* $L^2(\Omega\times Q)$

$$\begin{array}{rcl} u^{\Delta t} & \to & u_\epsilon, \\ h(u^{\Delta t}) & \to & h(u_\epsilon), \\ \psi_\epsilon(u^{\Delta t}) & \to & \psi_\epsilon(u_\epsilon), \\ w_s(u^{\Delta t}) & \to & w_s(u_\epsilon). \end{array}$$

Proof. By considering as previously the test function $v = u_{n+1}$ in (2.2), we have for any $n > 0$ after taking the expectation

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\left[\|u_{n+1} - u_n\|^2\right] + \frac{1}{2}\mathbb{E}\left[\|u_{n+1}\|^2\right] - \frac{1}{2}\mathbb{E}\left[\|u_n\|^2\right] + \Delta t\mathbb{E}\left[\|\nabla u_{n+1}\|^2\right] \\ &+\Delta t\mathbb{E}\left[\int_D \psi_\epsilon(u_{n+1})u_{n+1}dx\right] \\ \leqslant\ & \Delta t\mathbb{E}\left[\int_D (w_s(u_{n+1}) + f_n)\,u_{n+1}dx\right] + \frac{1}{2}\mathbb{E}\left[\|u_{n+1} - u_n\|^2\right] + \frac{\Delta t}{2}\mathbb{E}\left[\|h(u_n)\|^2\right], \end{aligned}$$

which yields, after multiplying by $e^{-cn\Delta t}$, for any positive c

$$\begin{aligned} &e^{-cn\Delta t}\mathbb{E}\left[\|u_{n+1}\|^2\right] - e^{-c(n-1)\Delta t}\mathbb{E}\left[\|u_n\|^2\right] + 2\Delta te^{-cn\Delta t}\mathbb{E}\left[\|\nabla u_{n+1}\|^2\right] \\ &+2\Delta te^{-cn\Delta t}\mathbb{E}\left[\int_D \psi_\epsilon(u_{n+1})u_{n+1}dx\right] \\ \leqslant\ & 2e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D (w_s(u_{n+1}) + f_n)\,u_{n+1}dx\right] + \Delta te^{-cn\Delta t}\mathbb{E}\left[\|h(u_n)\|^2\right] \\ &+ \left(e^{-cn\Delta t} - e^{-c(n-1)\Delta t}\right)\mathbb{E}\left[\|u_n\|^2\right]. \end{aligned}$$

For any $k > 0$, by summing this inequality from $n = 0$ to k, one gets

$$\begin{aligned} &e^{-ck\Delta t}\mathbb{E}\left[\|u_{k+1}\|^2\right] + 2\sum_{n=0}^{k} \Delta te^{-cn\Delta t}\mathbb{E}\left[\|\nabla u_{n+1}\|^2\right] + 2\sum_{n=0}^{k} \Delta te^{-cn\Delta t}\mathbb{E}\left[\int_D \psi_\epsilon(u_{n+1})u_{n+1}dx\right] \\ \leqslant\ & e^{c\Delta t}\|u_0\|^2 + 2\sum_{n=0}^{k} e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D (w_s(u_{n+1}) + f_n)\,u_{n+1}dx\right] + \sum_{n=0}^{k} \Delta te^{-cn\Delta t}\mathbb{E}\left[\|h(u_n)\|^2\right] \\ &+ \sum_{n=0}^{k}\left(e^{-cn\Delta t} - e^{-c(n-1)\Delta t}\right)\mathbb{E}\left[\|u_n\|^2\right]. \end{aligned} \tag{2.4}$$

Since ψ_ϵ is non-decreasing and satisfies $\psi_\epsilon(0) = 0$, we have

$$\begin{aligned}\mathbb{E}\left[\Delta t \sum_{n=0}^{k} e^{-cn\Delta t} \int_D \psi_\epsilon(u_{n+1})u_{n+1}dx\right] &\geqslant \mathbb{E}\left[\sum_{n=0}^{k} \int_{t_n}^{t_{n+1}} e^{-cs} \int_D \psi_\epsilon(u_{n+1})u_{n+1}dxds\right] \\ &= \mathbb{E}\left[\int_0^{t_{k+1}} e^{-cs} \int_D \psi_\epsilon(u^{\Delta t})u^{\Delta t}dxds\right],\end{aligned}$$

and in the same way

$$\sum_{n=0}^{k} \Delta t e^{-cn\Delta t}\mathbb{E}\left[\|\nabla u_{n+1}\|^2\right] \geqslant \int_0^{t_{k+1}} e^{-cs}\mathbb{E}\left[\|\nabla u^{\Delta t}\|^2\right]ds.$$

Moreover,

$$\begin{aligned}\sum_{n=0}^{k} \Delta t e^{-cn\Delta t}\mathbb{E}\left[\|h(u_n)\|^2\right] &= \Delta t\|h(u_0)\|^2 + \sum_{n=0}^{k-1} \Delta t e^{-c(n+1)\Delta t}\mathbb{E}\left[\|h(u_{n+1})\|^2\right] \\ &\leqslant \Delta t\|h(u_0)\|^2 + \sum_{n=0}^{k-1} \int_{t_n}^{t_{n+1}} e^{-cs}\mathbb{E}\left[\|h(u_{n+1})\|^2\right]ds \\ &= \Delta t\|h(u_0)\|^2 + \int_0^{t_k} e^{-cs}\mathbb{E}\left[\|h(u^{\Delta t})\|^2\right]ds.\end{aligned}$$

Besides,

$$\begin{aligned}\sum_{n=0}^{k} \left(e^{-cn\Delta t} - e^{-c(n-1)\Delta t}\right)\mathbb{E}\left[\|u_n\|^2\right] &= (1 - e^{c\Delta t})\|u_0\|^2 + \sum_{n=1}^{k} \left(e^{-cn\Delta t} - e^{-c(n-1)\Delta t}\right)\mathbb{E}\left[\|u_n\|^2\right] \\ &= (1 - e^{c\Delta t})\|u_0\|^2 - c\sum_{n=1}^{k} \left(\int_{t_{n-1}}^{t_n} e^{-cs}ds\right)\mathbb{E}\left[\|u_n\|^2\right] \\ &= (1 - e^{c\Delta t})\|u_0\|^2 - c\sum_{n=1}^{k} \left(\int_{t_{n-1}}^{t_n} e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds\right) \\ &\leqslant (1 - e^{c\Delta t})\|u_0\|^2 - ce^{-c\Delta t}\int_0^{t_k} e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds.\end{aligned}$$

In this way, we can write inequality (2.4) in the following manner

$$\begin{aligned}&e^{-ck\Delta t}\mathbb{E}\left[\|u_{k+1}\|^2\right] + 2\int_0^{t_{k+1}} e^{-cs}\mathbb{E}\left[\|\nabla u^{\Delta t}\|^2\right]ds + 2\int_0^{t_{k+1}} e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u^{\Delta t})u^{\Delta t}dx\right]ds \\ \leqslant\ &e^{c\Delta t}\|u_0\|^2 + 2\sum_{n=0}^{k} e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D (w_s(u_{n+1}) + f_n)\, u_{n+1}dx\right] \\ &+\Delta t\|h(u_0)\|^2 + \int_0^{t_k} e^{-cs}\mathbb{E}\left[\|h(u^{\Delta t})\|^2\right]ds \\ &+(1 - e^{c\Delta t})\|u_0\|^2 - ce^{-c\Delta t}\int_0^{t_k} e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds.\end{aligned}$$

For $t \in [t_k, t_{k+1})$, we have since $e^{-ck\Delta t} \geqslant e^{-ct}$ and $(t - \Delta t)^+ \leqslant t_k$

$$\begin{aligned}&e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right] + 2\int_0^{t} e^{-cs}\mathbb{E}\left[\|\nabla u^{\Delta t}\|^2\right]ds + 2\int_0^{t} e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u^{\Delta t})u^{\Delta t}dx\right]ds \\ \leqslant\ &\|u_0\|^2 + 2\sum_{n=0}^{k} e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D (w_s(u_{n+1}) + f_n)\, u_{n+1}dx\right] \\ &+\Delta t\|h(u_0)\|^2 + \int_0^{t} e^{-cs}\mathbb{E}\left[\|h(u^{\Delta t})\|^2\right]ds \\ &-ce^{-c\Delta t}\int_0^{(t-\Delta t)^+} e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds.\end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned}
\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla u^{\Delta t}\|^2\right]ds &= \int_0^t e^{-cs}\mathbb{E}\left[\|\nabla(u^{\Delta t}-u_\epsilon)\|^2 + 2\int_D \nabla u^{\Delta t}\nabla u_\epsilon dx - \|\nabla u_\epsilon\|^2\right]ds,\\
\int_0^t e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u^{\Delta t})u^{\Delta t}dx\right]ds &= \int_0^t e^{-cs}\mathbb{E}\left[\int_D \left(\psi_\epsilon(u^{\Delta t})-\psi_\epsilon(u_\epsilon)\right)(u^{\Delta t}-u_\epsilon)dx\right]ds\\
&\quad + \int_0^t e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u_\epsilon)(u^{\Delta t}-u_\epsilon)+\psi_\epsilon(u^{\Delta t})u_\epsilon dx\right]ds,
\end{aligned}$$

and

$$\int_0^t e^{-cs}\mathbb{E}\left[\|h(u^{\Delta t})\|^2\right]ds = \int_0^t e^{-cs}\mathbb{E}\left[\|h(u^{\Delta t})-h(u_\epsilon)\|^2 + 2\int_D h(u^{\Delta t})h(u_\epsilon)dx - \|h(u_\epsilon)\|^2\right]ds.$$

Besides, there exists a constant $C>0$ independent of Δt such that

$$\begin{aligned}
&\left|\sum_{n=0}^k e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D w_s(u_{n+1})u_{n+1}dx\right] - \int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u^{\Delta t}dx\right]ds\right|\\
\le\ & \left|\sum_{n=0}^k e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D w_s(u_{n+1})u_{n+1}dx\right] - \int_0^{t_{k+1}} e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u^{\Delta t}dx\right]ds\right|\\
&+\left|\int_t^{t_{k+1}} e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u^{\Delta t}dx\right]ds\right|\\
\le\ & \left|\sum_{n=0}^k \int_{t_n}^{t_{n+1}}\left(e^{-cn\Delta t}-e^{-cs}\right)\mathbb{E}\left[\int_D w_s(u_{n+1})u_{n+1}dx\right]ds\right| + CL_s\Delta t\sup_k\mathbb{E}\left[\|u_k\|^2_{L^2(D)}\right]\\
\leqslant\ & c\sum_{n=0}^k\int_{t_n}^{t_{n+1}}\Delta t\mathbb{E}\left[\int_D |w_s(u_{n+1})u_{n+1}|dx\right]ds + CL_s\Delta t\\
\leqslant\ & c\Delta t L_s\|u^{\Delta t}\|^2_{L^2(\Omega\times Q)} + CL_s\Delta t\\
\leqslant\ & C\Delta t,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left|\sum_{n=0}^k e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D f_n u_{n+1}dx\right] - \int_0^t e^{-cs}\mathbb{E}\left[\int_D f_{\Delta t}u^{\Delta t}dx\right]ds\right|\\
\le\ & \left|\sum_{n=0}^k e^{-cn\Delta t}\Delta t\mathbb{E}\left[\int_D f_n u_{n+1}dx\right] - \int_0^{t_{k+1}} e^{-cs}\mathbb{E}\left[\int_D f_{\Delta t}u^{\Delta t}dx\right]ds\right|\\
&+\left|\int_t^{t_{k+1}} e^{-cs}\mathbb{E}\left[\int_D f_{\Delta t}u^{\Delta t}dx\right]ds\right|\\
=\ & \left|\sum_{n=0}^k\int_{t_n}^{t_{n+1}}\left(e^{-cn\Delta t}-e^{-cs}\right)\mathbb{E}\left[\int_D f_n u_{n+1}dx\right]ds\right| + \left|\int_t^{t_{k+1}} e^{-cs}\mathbb{E}\left[\int_D f_k u_{k+1}dx\right]ds\right|\\
\leqslant\ & c\Delta t\|f_{\Delta t}\|_{L^2(\Omega\times Q)}\|u^{\Delta t}\|_{L^2(\Omega\times Q)} + \int_t^{t_{k+1}}\|f_k\|_{L^2(\Omega\times D)}\|u_{k+1}\|_{L^2(\Omega\times D)}ds\\
\leqslant\ & c\Delta t\|f_{\Delta t}\|_{L^2(\Omega\times Q)}\|u^{\Delta t}\|_{L^2(\Omega\times Q)} + \|u^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))}\sqrt{\Delta t}\Big(\int_t^{t_{k+1}}\|f_k\|^2_{L^2(\Omega\times D)}ds\Big)^{1/2}\\
\leqslant\ & C\Delta t + \tilde\epsilon(\Delta t),\qquad \text{where } \tilde\epsilon(\Delta t)\to 0 \text{ when } \Delta t\to 0,
\end{aligned}$$

as well as

$$\begin{aligned}
&ce^{-c\Delta t}\int_0^{(t-\Delta t)^+} e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds\\
=\ & ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds - ce^{-c\Delta t}\int_{(t-\Delta t)^+}^t e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds\\
\geqslant\ & ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\left[\|u^{\Delta t}\|^2\right]ds - ce^{-c\Delta t}\Delta t\|u^{\Delta t}\|^2_{L^\infty(0,T;L^2(\Omega\times D))}\\
=\ & ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\left[\|u^{\Delta t}-u_\epsilon\|^2 + 2\int_D u^{\Delta t}u_\epsilon dx - \|u_\epsilon\|^2\right]ds - C\Delta t.
\end{aligned}$$

Using these we can write inequality (2.5) in the following way :

$$\begin{aligned}
& e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right] + 2\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla(u^{\Delta t}-u_\epsilon)\|^2 + 2\int_D \nabla u^{\Delta t}\nabla u_\epsilon dx - \|\nabla u_\epsilon\|^2\right]ds \\
& +2\int_0^t e^{-cs}\mathbb{E}\left[\int_D \left(\psi_\epsilon(u^{\Delta t}) - \psi_\epsilon(u_\epsilon)\right)(u^{\Delta t}-u_\epsilon)dx\right]ds \\
& +2\int_0^t e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u_\epsilon)(u^{\Delta t}-u_\epsilon) + \psi_\epsilon(u^{\Delta t})u_\epsilon dx\right]ds \\
\leqslant\ & \|u_0\|^2 + 2\int_0^t e^{-cs}\mathbb{E}\left[\int_D \left(w_s(u^{\Delta t}) + f_{\Delta t}\right)u^{\Delta t}dx\right]ds + \Delta t\|h(u_0)\|^2 \\
& +\int_0^t e^{-cs}\mathbb{E}\left[\|h(u^{\Delta t}) - h(u_\epsilon)\|^2 + 2\int_D h(u^{\Delta t})h(u_\epsilon)dx - \|h(u_\epsilon)\|^2\right]ds \\
& -ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\left[\|u^{\Delta t}-u_\epsilon\|^2 + 2\int_D u^{\Delta t}u_\epsilon dx - \|u_\epsilon\|^2\right]ds + C\Delta t + \tilde{\epsilon}(\Delta t).
\end{aligned}$$

And so by noting that

$$\begin{aligned}
\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u^{\Delta t}dx\right]ds \quad = \quad & \int_0^t e^{-cs}\mathbb{E}\left[\int_D \left(w_s(u^{\Delta t}) - w_s(u_\epsilon)\right)(u^{\Delta t}-u_\epsilon)dx\right]ds \\
& +\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u_\epsilon)(u^{\Delta t}-u_\epsilon)dx\right]ds \\
& +\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u_\epsilon dx\right]ds,
\end{aligned}$$

we obtain

$$\begin{aligned}
& e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right] + 2\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla(u^{\Delta t}-u_\epsilon)\|^2 + 2\int_D \nabla u^{\Delta t}\nabla u_\epsilon dx - \|\nabla u_\epsilon\|^2\right]ds \\
& +2\int_0^t e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u_\epsilon)(u^{\Delta t}-u_\epsilon) + \psi_\epsilon(u^{\Delta t})u_\epsilon dx\right]ds \\
\leqslant\ & \|u_0\|^2 + 2\int_0^t e^{-cs}\mathbb{E}\left[\int_D f_{\Delta t}u^{\Delta t}dx\right]ds + 2\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u_\epsilon dx\right]ds \\
& +2\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u_\epsilon)(u^{\Delta t}-u_\epsilon)dx\right]ds + \Delta t\|h(u_0)\|^2 \\
& +\int_0^t e^{-cs}\mathbb{E}\left[2\int_D h(u^{\Delta t})h(u_\epsilon)dx - \|h(u_\epsilon)\|^2\right]ds \\
& -ce^{-c\Delta t}\int_0^t e^{-cs}\mathbb{E}\left[2\int_D u^{\Delta t}u_\epsilon dx - \|u_\epsilon\|^2\right]ds \\
& +C\Delta t + \left(C_h^2 + 2L_s - ce^{-c\Delta t} + \frac{2}{\epsilon}\right)\|u^{\Delta t}-u_\epsilon\|^2_{L^2(\Omega\times Q)} + \tilde{\epsilon}(\Delta t).
\end{aligned}$$

By choosing c such that $C_h^2 + 2L_s - ce^{-c\Delta t} + \frac{2}{\epsilon} \leqslant 0$ we finally get

$$\begin{aligned}
& \int_0^T e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right]dt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D \nabla u^{\Delta t}\nabla u_\epsilon dx - \|\nabla u_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u_\epsilon)(u^{\Delta t}-u_\epsilon) + \psi_\epsilon(u^{\Delta t})u_\epsilon dx\right]dsdt \\
\leqslant\ & T\|u_0\|^2 + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D f_{\Delta t}u^{\Delta t}dx\right]dsdt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u_\epsilon dx\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u_\epsilon)(u^{\Delta t}-u_\epsilon)dx\right]dsdt + T\Delta t\|h(u_0)\|^2 \\
& +\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D h(u^{\Delta t})h(u_\epsilon)dx - \|h(u_\epsilon)\|^2\right]dsdt \\
& -ce^{-c\Delta t}\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D u^{\Delta t}u_\epsilon dx - \|u_\epsilon\|^2\right]dsdt + CT\Delta t + \tilde{\epsilon}(\Delta t). \qquad (2.6)
\end{aligned}$$

Now, by applying the Itô formula to the process u_ϵ and the function $F(t,v) = e^{-ct}\|v\|^2$ we get the following energy equality ([15] Theorem 4.17 p.105, [18] Theorem 4.2.5 p.75)

$$\begin{aligned}
& \int_0^T e^{-ct}\mathbb{E}\left[\|u_\epsilon(t)\|^2\right]dt + c\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|u_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla u_\epsilon\|^2\right]dsdt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D \chi_\epsilon u_\epsilon dx\right]dsdt \\
= \; & T\|u_0\|^2 + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D (w_\epsilon + f)u_\epsilon dx\right]dsdt + \int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|h_\epsilon\|^2\right]dsdt.
\end{aligned}$$

By replacing $T\|u_0\|^2$ in (2.6) using this last equality, we obtain

$$\begin{aligned}
& \int_0^T e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right]dt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D \nabla u^{\Delta t}\nabla u_\epsilon dx - \|\nabla u_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D \psi_\epsilon(u_\epsilon)(u^{\Delta t} - u_\epsilon) + \psi_\epsilon(u^{\Delta t})u_\epsilon dx\right]dsdt \\
\leqslant \; & \int_0^T e^{-ct}\mathbb{E}\left[\|u_\epsilon(t)\|^2\right]dt + c\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|u_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla u_\epsilon\|^2\right]dsdt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D \chi_\epsilon u_\epsilon dx\right]dsdt \\
& -2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D (w_\epsilon + f)u_\epsilon dx\right]dsdt - \int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D h_\epsilon^2 dx\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D f_{\Delta t}u^{\Delta t}dx\right]dsdt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u^{\Delta t})u_\epsilon dx\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_s(u_\epsilon)(u^{\Delta t} - u_\epsilon)dx\right]dsdt + T\Delta t\|h(u_0)\|^2 \\
& +\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D h(u^{\Delta t})h(u_\epsilon)dx - \|h(u_\epsilon)\|^2\right]dsdt \\
& -ce^{-c\Delta t}\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D u^{\Delta t}u_\epsilon dx - \|u_\epsilon\|^2\right]dsdt + CT\Delta t + \tilde{\epsilon}(\Delta t).
\end{aligned}$$

And by passing to the superior limit we get since $f_{\Delta t} \underset{\Delta t}{\longrightarrow} f$ in $L^2(\Omega \times Q)$

$$\begin{aligned}
& \limsup_{\Delta t}\int_0^T e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right]dt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla u_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D \chi_\epsilon u_\epsilon dx\right]dsdt \\
\leqslant \; & \int_0^T e^{-ct}\mathbb{E}\left[\|u_\epsilon(t)\|^2\right]dt + c\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|u_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla u_\epsilon\|^2\right]dsdt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D \chi_\epsilon u_\epsilon dx\right]dsdt \\
& -2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D (w_\epsilon + f)u_\epsilon dx\right]dsdt - \int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|h_\epsilon\|^2\right]dsdt \\
& +2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D f u_\epsilon dx\right]dsdt + 2\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\int_D w_\epsilon u_\epsilon dx\right]dsdt \\
& +\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[2\int_D h_\epsilon h(u_\epsilon)dx - \|h(u_\epsilon)\|^2\right]dsdt - c\int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|u_\epsilon\|^2\right]dsdt.
\end{aligned}$$

Finally

$$\limsup_{\Delta t}\int_0^T e^{-ct}\mathbb{E}\left[\|u^{\Delta t}(t)\|^2\right]dt \quad \leqslant \quad \int_0^T e^{-ct}\mathbb{E}\left[\|u_\epsilon(t)\|^2\right]dt - \int_0^T\int_0^t e^{-cs}\mathbb{E}\left[\|h(u_\epsilon) - h_\epsilon\|^2\right]dsdt$$

Thus $u^{\Delta t} \to u_\epsilon$ in $L^2(\Omega \times Q)$, $h(u_\epsilon) = h_\epsilon$, $\chi_\epsilon = \psi_\epsilon(u_\epsilon)$ and $w_\epsilon = w_s(u_\epsilon)$. ■

2.2 Uniqueness of u_ϵ

Proposition 2.6 *There exists a unique solution to Problem* (2.1).

Proof. Let $u_\epsilon, \tilde{u}_\epsilon$ be two solutions of Problem (2.1) associated with the same initial condition u_0. Then, $u_\epsilon - \tilde{u}_\epsilon$ satisfies

$$d(u_\epsilon - \tilde{u}_\epsilon) + \big[\psi_\epsilon(u_\epsilon) - \psi_\epsilon(\tilde{u}_\epsilon) - \Delta(u_\epsilon - \tilde{u}_\epsilon)\big]dt = \big[w_s(u_\epsilon) - w_s(\tilde{u}_\epsilon)\big]dt + \big[h(u_\epsilon) - h(\tilde{u}_\epsilon)\big]dW.$$

Let c be a positive constant, by applying the Itô formula [15, 18]), to the process $u_\epsilon - \tilde{u}_\epsilon$ and the function $F(t,v) = e^{-ct}\|v\|^2$, one gets for any $t \in [0,T]$

$$\begin{aligned}
&e^{-ct}\mathbb{E}\left[\|(u_\epsilon - \tilde{u}_\epsilon)(t)\|^2\right] + c\int_0^t e^{-cs}\|u_\epsilon - \tilde{u}_\epsilon\|^2 ds + 2\int_0^t e^{-cs}\mathbb{E}\left[\|\nabla(u_\epsilon - \tilde{u}_\epsilon)\|^2\right] ds \\
&+2\int_0^t e^{-cs}\mathbb{E}\big[(\psi_\epsilon(u_\epsilon) - \psi_\epsilon(\tilde{u}_\epsilon))(u_\epsilon - \tilde{u}_\epsilon)\big]ds \\
= \ & 2\int_0^t e^{-cs}\mathbb{E}\big[(w_s(u_\epsilon) - w_s(\tilde{u}_\epsilon))(u_\epsilon - \tilde{u}_\epsilon)\big]ds + \int_0^t e^{-cs}\mathbb{E}\left[\|h(u_\epsilon) - h(\tilde{u}_\epsilon)\|^2\right] ds.
\end{aligned}$$

In this way, by choosing c such that $2L_s + C_h^2 - c < 0$, one gets

$$\forall t \in I, \quad e^{-ct}\mathbb{E}\left[\|(u_\epsilon - \tilde{u}_\epsilon)(t)\|^2\right] \leqslant 0,$$

which proves the announced uniqueness result. ■

3 Proof of the main result

The aim of this section is to prove our main result, Theorem 1.2. To do this, firstly we show the following convergence result.

Proposition 3.1 *Up to subsequences,* $(u_\epsilon)_{\epsilon>0}$ *converges strongly in* $\mathcal{N}_w^2\left(0,T,H^1(D)\right)$ *and weakly* in* $L^\infty\left(0,T;L^2(\Omega\times D)\right)$ *to a function* u*,* $(\psi_\epsilon(u_\epsilon))_{\epsilon>0}$ *converges weakly in* $\mathcal{N}_w^2\left(0,T,L^2(D)\right)$ *to a function* $\xi \in \partial I_{[0,1]}(u)$*, and in addition almost everywhere in* $(0,T)$*,* $\mathbb{P}$*-almost surely in* Ω *and for any* v *in* $H^1(D)$

$$< \partial_t\left(u - \int_0^\cdot h(u)dW\right), v >_{V',V} + \int_D \nabla u.\nabla v dx + \int_D \xi v dx = \int_D (w_s(u) + f)\, v dx.$$

3.1 Study of the sequences $(u_\epsilon)_{\epsilon>0}$ and $(\psi_\epsilon(u_\epsilon))_{\epsilon>0}$

Proposition 3.2 $(u_\epsilon)_{\epsilon>0}$ *is bounded in* $L^\infty\left(0,T;L^2(\Omega\times D)\right) \cap \mathcal{N}_w^2\left(0,T,H^1(D)\right)$.

Proof. This first result is a direct consequence of Proposition 2.2 and of the lower semi-continuity of the norm for the weak, or weak-*, convergence. ■

Proposition 3.3 $(\psi_\epsilon(u_\epsilon))_{\epsilon>0}$ *is bounded in* $\mathcal{N}_w^2\left(0,T,L^2(D)\right)$.

Proof. We consider the convex antiderivative ϕ_ϵ of ψ_ϵ defined by

$$\phi_\epsilon(v) = \begin{cases} \dfrac{v^2}{2\epsilon} & \text{if} \quad v \leqslant 0 \\ 0 & \text{if} \quad v \in [0,1] \\ \dfrac{(v-1)^2}{2\epsilon} & \text{if} \quad v \geqslant 1. \end{cases}$$

By applying the Itô formula ([15, 18]) to the process u_ϵ and the function $F(t,v) = \int_D \phi_\epsilon(v)dx$, one gets for any $t \in [0,T]$

$$\begin{aligned}
\int_D \phi_\epsilon(u_\epsilon(t))dx = &\int_D \phi_\epsilon(u_0)dx + \int_0^t \int_D \psi_\epsilon(u_\epsilon)\big\{\Delta u_\epsilon - \psi_\epsilon(u_\epsilon) + w_s(u_\epsilon) + f\big\}dxds \\
&+ \int_0^t \int_D \psi_\epsilon(u_\epsilon)h(u_\epsilon)dxdW(s) + \frac{1}{2}\int_0^t \int_D \psi'_\epsilon(u_\epsilon)h^2(u_\epsilon)dxds. \qquad (3.1)
\end{aligned}$$

Since $h(0) = h(1) = 0$ one gets

$$\begin{aligned}
&\int_0^t \int_D \psi_\epsilon'(u_\epsilon) h^2(u_\epsilon) dx ds \\
&= \int_0^t \int_{u_\epsilon<0} \psi_\epsilon'(u_\epsilon) h^2(u_\epsilon) dx ds + \int_0^t \int_{u_\epsilon>1} \psi_\epsilon'(u_\epsilon) h^2(u_\epsilon) dx ds \\
&= \int_0^t \int_{u_\epsilon<0} \psi_\epsilon'(u_\epsilon) \left[h(u_\epsilon) - h(0)\right]^2 dx ds + \int_0^t \int_{u_\epsilon>1} \psi_\epsilon'(u_\epsilon) \left[h(u_\epsilon) - h(1)\right]^2 dx ds \\
&\leqslant C_h^2 \int_0^t \int_{u_\epsilon<0} \psi_\epsilon'(u_\epsilon) u_\epsilon^2 dx ds + C_h^2 \int_0^t \int_{u_\epsilon>1} \psi_\epsilon'(u_\epsilon)(u_\epsilon - 1)^2 dx ds \\
&\leqslant 2C_h^2 \int_0^t \int_D \phi_\epsilon(u_\epsilon(s)) dx ds.
\end{aligned}$$

After taking the expectation in (3.1) one gets since $\phi_\epsilon(u_0) = 0$

$$\begin{aligned}
&\mathbb{E}\left[\int_D \phi_\epsilon(u_\epsilon(t)) dx\right] + \mathbb{E}\left[\int_0^t \int_D \psi_\epsilon'(u_\epsilon) |\nabla u_\epsilon|^2 dx ds\right] + \|\psi_\epsilon(u_\epsilon)\|^2_{L^2(\Omega\times Q)} \\
\leqslant\ & \frac{1}{2}\|\psi_\epsilon(u_\epsilon)\|^2_{L^2(\Omega\times Q)} + \|w_s(u_\epsilon)\|^2_{L^2(\Omega\times Q)} + \|f\|^2_{L^2(\Omega\times Q)} + C_h^2 \mathbb{E}\left[\int_0^t \int_D \phi_\epsilon(u_\epsilon(s)) dx ds\right],
\end{aligned}$$

and using the fact that $\psi_\epsilon' \geqslant 0$, one gets

$$\begin{aligned}
&\mathbb{E}\left[\int_D \phi_\epsilon(u_\epsilon(t)) dx\right] + \frac{1}{2}\|\psi_\epsilon(u_\epsilon)\|^2_{L^2(\Omega\times Q)} \\
\leqslant\ & \|w_s(u_\epsilon)\|^2_{L^2(\Omega\times Q)} + \|f\|^2_{L^2(\Omega\times Q)} + 2C_h^2 \mathbb{E}\left[\int_0^t \int_D \phi_\epsilon(u_\epsilon(s)) dx ds\right].
\end{aligned}$$

Since $\phi_\epsilon(u_\epsilon(t)) \geqslant 0$, by using Gronwall's lemma one gets the existence of $\tilde{C} > 0$ independent of ϵ such that

$$2C_h^2 \mathbb{E}\left[\int_0^t \int_D \phi_\epsilon(u_\epsilon(s)) dx ds\right] \leqslant \tilde{C},$$

thus

$$\|\psi_\epsilon(u_\epsilon)\|^2_{L^2(\Omega\times Q)} \quad \leqslant \quad 2L_s^2 \|u_\epsilon\|^2_{L^2(\Omega\times Q)} + 2\|f\|^2_{L^2(\Omega\times Q)} + 2C,$$

and then $(\psi_\epsilon(u_\epsilon))_{\epsilon>0}$ is bounded in $\mathcal{N}_w^2\left(0, T, L^2(D)\right)$ thanks to Proposition 3.2. ■

Using this result, the convergence of $u^{\Delta t}$ to u_ϵ in $L^2(\Omega \times Q)$ and Proposition 2.3, one gets the following bounds, in particular for the sequences $\left(\partial_t \left(u_\epsilon - \int_0^\cdot h(u_\epsilon) dW\right)\right)_{\epsilon>0}$, $\left(\frac{u_\epsilon^-}{\epsilon}\right)_{\epsilon>0}$ and $\left(\frac{(u_\epsilon - 1)^+}{\epsilon}\right)_{\epsilon>0}$.

Proposition 3.4 *There exists a constant $C > 0$ which only depends on T, C_h, u_0, L_s and f such that*

$$\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega;H^1(D)))},\ \|\partial_t(u_\epsilon - \int_0^\cdot h(u_\epsilon) dW)\|_{L^2(\Omega\times Q)},\ \|\Delta u_\epsilon\|_{L^2(\Omega\times Q)} \leqslant C.$$

Proof. The boundedness of $\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega;H^1(D)))}$ is a direct consequence of Proposition 2.3 and the lower semi-continuity of the norm for the weak-* convergence. Moreover, thanks to Proposition 2.3, there exists a constant $C > 0$ which only depends on T, C_h, u_0, L_s and f such that

$$\|\partial_t(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega\times Q)} \quad \leqslant \quad C\left(1 + \|\psi_\epsilon(u^{\Delta t})\|_{L^2(\Omega\times Q)}\right).$$

Then, using the lower semi-continuity of the norm for the weak convergence, the strong convergence of $\psi_\epsilon(u^{\Delta t})$ towards $\psi_\epsilon(u_\epsilon)$ in $L^2(\Omega\times Q)$ given by Proposition 2.5, and Proposition 3.3, there exists $\tilde{C}$ which only depends on T, C_h, u_0, L_s and f such that

$$\begin{aligned}
\|\partial_t\Big(u_\epsilon - \int_0^\cdot h(u_\epsilon) dW\Big)\|_{L^2(\Omega\times Q)} &\leqslant \liminf_{\Delta t} \|\partial_t\left(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}\right)\|_{L^2(\Omega\times Q)} \\
&\leqslant C\left(1 + \liminf_{\Delta t}\left\{\|\psi_\epsilon(u^{\Delta t}) - \psi_\epsilon(u_\epsilon)\|_{L^2(\Omega\times Q)} + \|\psi_\epsilon(u_\epsilon)\|_{L^2(\Omega\times Q)}\right\}\right) \\
&\leqslant \tilde{C}.
\end{aligned}$$

Finally, a comparison in (2.1a) gives the boundedness of Δu_ϵ in $L^2(\Omega\times Q)$. ∎

Proposition 3.5 *$\left(\frac{u_\epsilon^-}{\epsilon}\right)_{\epsilon>0}$ and $\left(\frac{(u_\epsilon-1)^+}{\epsilon}\right)_{\epsilon>0}$ are bounded in $L^2(\Omega\times Q)$.*

Proof. Since $\psi_\epsilon(u_\epsilon) = -\frac{u_\epsilon^-}{\epsilon} + \frac{(u_\epsilon-1)^+}{\epsilon}$ is bounded in $L^2(\Omega\times Q)$ and $u_\epsilon^-(u_\epsilon-1)^+ = 0$, one gets that

$$\left\|-\frac{u_\epsilon^-}{\epsilon}\right\|^2_{L^2(\Omega\times Q)} + \left\|\frac{(u_\epsilon-1)^+}{\epsilon}\right\|^2_{L^2(\Omega\times Q)} = \left\|-\frac{u_\epsilon^-}{\epsilon} + \frac{(u_\epsilon-1)^+}{\epsilon}\right\|^2_{L^2(\Omega\times Q)} = \|\psi_\epsilon(u_\epsilon)\|^2_{L^2(\Omega\times Q)} \leqslant C,$$

and the result holds. ∎

3.2 Existence of a solution

Proposition 3.6 *There exist $u\in\mathcal{N}^2_w\big(0,T,H^1(D)\big)\cap L^\infty\left(0,T,L^2\left(\Omega,H^1(D)\right)\right)\cap L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)$, ξ,ξ_1,ξ_2 in $\mathcal{N}^2_w\big(0,T,L^2(D)\big)$ and $\overline{w}_s,\overline{h}$ in $\mathcal{N}^2_w\big(0,T,H^1(D)\big)$ such that up to subsequences denoted in the same way, the following convergences hold as $\epsilon\to 0$*

$$\begin{aligned}
u_\epsilon &\rightharpoonup u \text{ in } \mathcal{N}^2_w\left(0,T;H^1(D)\right),\\
u_\epsilon &\overset{*}{\rightharpoonup} u \text{ in } L^\infty\big(0,T,L^2(\Omega,H^1(D))\big),\\
h(u_\epsilon) &\rightharpoonup \overline{h} \text{ in } L^2\left(\Omega\times(0,T),H^1(D)\right),\\
u_\epsilon - \int_0^\cdot h(u_\epsilon)dW &\rightharpoonup u - \int_0^\cdot \overline{h}dW \text{ in } L^2\left(\Omega,H^1(Q)\right),\\
w_s(u_\epsilon) &\rightharpoonup \overline{w}_s \text{ in } L^2\left(\Omega\times(0,T),H^1(D)\right),\\
\psi_\epsilon(u_\epsilon) &\rightharpoonup \xi \text{ in } L^2(\Omega\times Q),\\
-\frac{u_\epsilon^-}{\epsilon} &\rightharpoonup \xi_1 \text{ in } L^2(\Omega\times Q), \text{ with } \xi_1\leqslant 0,\\
\frac{(u_\epsilon-1)^+}{\epsilon} &\rightharpoonup \xi_2 \text{ in } L^2(\Omega\times Q), \text{ with } \xi_2\geqslant 0.
\end{aligned}$$

Proof. These results are immediate using estimates given by Proposition 3.2, 3.3, 3.4 and 3.5. Let us precise briefly that the regularity $u\in L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)$ comes from the fact that

$$u-\int_0^\cdot \overline{h}dW\in L^2\left(\Omega,H^1(Q)\right)\hookrightarrow L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right),$$

and the result holds since $\int_0^\cdot \overline{h}dW\in L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)$ ([15] Theorem 4.12 p.101).
Note also that $\xi_1\le 0$ (resp. $\xi_2\ge 0$) since $-\frac{u_\epsilon^-}{\epsilon}\le 0$ (resp. $\frac{(u_\epsilon-1)^+}{\epsilon}\ge 0$) and as the subset of the non-negative elements of $L^2(\Omega\times Q)$ is a closed convex set. ∎

Remark 3.7 *Note that $\psi_\epsilon(u_\epsilon) = -\frac{u_\epsilon^-}{\epsilon} + \frac{(u_\epsilon-1)^+}{\epsilon}$, and so $\xi=\xi_1+\xi_2$.*

Proposition 3.8 *Up to a subsequence $u_\epsilon(T)\rightharpoonup u(T)$ in $L^2(\Omega\times D)$.*

Proof. Since $u_\epsilon - \int_0^\cdot h(u_\epsilon)dW$ converges weakly to $u-\int_0^\cdot \overline{h}dW$ in $L^2\left(\Omega,H^1(Q)\right)$, it converges weakly in

$$L^2\left(\Omega,\mathscr{C}\left(0,T;L^2(D)\right)\right)\hookrightarrow\mathscr{C}\left(0,T;L^2(\Omega\times D)\right).$$

Moreover, using the linear continuity of the stochastic integral we have

$$\int_0^T h(u_\epsilon)dW \rightharpoonup \int_0^T \overline{h}dW \text{ in } L^2(\Omega\times D).$$

We deduce that $u_\epsilon(T)\rightharpoonup u(T)$ in $L^2(\Omega\times D)$. ∎

Remark 4 *In the same manner, since $u_\epsilon(.,0) = u_0$, one shows that $u(.,0) = u_0$ in the sense that*

$$\mathbb{P}\text{-}a.s \text{ in } \Omega,\ u(.,0) = \lim_{t\to 0} u(.,t) \text{ in } L^2(D).$$

Proposition 3.9 *$u_\epsilon^- \to 0$ and $(u_\epsilon - 1)^+ \to 0$ in $L^2(\Omega\times Q)$ as $\epsilon \to 0$.*

Proof. This result is a direct consequence of Proposition 3.5. Since there exists $C > 0$ such that

$$\left\| -\frac{u_\epsilon^-}{\epsilon}\right\|^2_{L^2(\Omega\times Q)} + \left\|\frac{(u_\epsilon - 1)^+}{\epsilon}\right\|^2_{L^2(\Omega\times Q)} \leqslant C$$

one gets

$$\|u_\epsilon^-\|^2_{L^2(\Omega\times Q)} + \|(u_\epsilon - 1)^+\|^2_{L^2(\Omega\times Q)} \leqslant C\epsilon^2,$$

and the result holds. ■

Remark 3.10 *Since*

$$\psi_\epsilon(u_\epsilon)u_\epsilon = \frac{u_\epsilon^-}{\epsilon} \times u_\epsilon^- + \frac{(u_\epsilon - 1)^+}{\epsilon} \times \Big((u_\epsilon - 1)^+ + 1\Big),$$

one gets that

$$\mathbb{E}\left[\int_Q \psi_\epsilon(u_\epsilon)u_\epsilon dxds\right] \to \mathbb{E}\left[\int_Q \xi_2 dxds\right] \text{ as } \epsilon \to 0.$$

By passing to the limit in the variational formulation (2.3), one gets that for any v in $H^1(D)$, α in $L^2(0,T)$ and any β in $L^2(\Omega)$

$$\int_\Omega \int_0^T \int_D \partial_t \left(u - \int_0^{\cdot} \overline{h}(s)dW(s)\right) v dx\alpha\beta dt d\mathbb{P} + \int_{\Omega\times Q} (\nabla u.\nabla v)\, \alpha\beta dxdtd\mathbb{P}$$
$$+ \int_{\Omega\times Q} \xi v\alpha\beta dxdtd\mathbb{P} = \int_{\Omega\times Q} (\overline{w}_s + f)\, v\alpha\beta dxdtd\mathbb{P}.$$

Since $H^1(D)$ is separable, one gets that almost everywhere in $(0,T)$, $\mathbb{P}$-almost surely in Ω and for any v in $H^1(D)$

$$\int_D \partial_t \left(u - \int_0^{\cdot} \overline{h}(s)dW(s)\right) v dx + \int_D \nabla u.\nabla v dx + \int_D \xi v dx = \int_D (\overline{w}_s + f)\, v dx.$$

In particular, u is a continuous $L^2(D)$-valued predictable process satisfying

$$u(t) = u(0) + \int_0^t (\Delta u - \xi + \overline{w}_s + f)\, ds + \int_0^t \overline{h}(s)dW(s),$$

where Δ denotes the Laplace operator on $H^1(D)$ associated with the formal Neumann boundary condition. Note that in our situation Δu is an element of $L^2(D)$, ∇u is then an element of $H(div, D)$ and the normal trace $\nabla u.\mathbf{n}$ exists in $H^{-1/2}(\partial D)$. Moreover, the solution u satisfies the following energy equality $\mathbb{P}$-almost surely in Ω and for any $t \in [0,T]$ by denoting as previously $\|.\| = \|.\|_{L^2(D)}$ and since $u(0) = u_0$

$$\mathbb{E}\left[\|u(t)\|^2\right] = \|u_0\|^2 - 2\mathbb{E}\left[\int_0^t \|\nabla u\|^2 ds\right] + 2\mathbb{E}\left[\int_0^t \int_D u(f - \xi + \overline{w}_s) dxds\right] + \mathbb{E}\left[\int_0^t \|\overline{h}\|^2 ds\right].$$

Proposition 3.11 *$\xi \in \partial I_{[0,1]}(u)$, $\overline{h} = h(u)$, $\overline{w}_s = w_s(u)$ and the convergence of u_ϵ towards u holds strongly in $L^2\big((0,T)\times\Omega, H^1(D)\big)$.*

Proof. We use here the same type of arguments as in the proof of Proposition 2.5. By applying the Itô formula to the process u_ϵ and the function $F(t,v) = \frac{1}{2}e^{-\alpha t}\|v\|^2$ with $\alpha > 0$, one gets that

$$e^{-\alpha T}\frac{1}{2}\|u_\epsilon(T)\|^2 - \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\int_Q \alpha e^{-\alpha s}|u_\epsilon|^2 dxds + \int_Q e^{-\alpha s}|\nabla u_\epsilon|^2 dxds + \int_Q e^{-\alpha s}u_\epsilon\psi_\epsilon(u_\epsilon)dxds$$
$$= \int_Q e^{-\alpha s}u_\epsilon\big(w_s(u_\epsilon) + f\big)dxds + \int_0^T \int_D e^{-\alpha s}u_\epsilon h(u_\epsilon)dxdW(s) + \frac{1}{2}\int_Q e^{-\alpha s}h^2(u_\epsilon)dxds.$$

Thus by taking the expectation

$$e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]-\frac{1}{2}\|u_0\|^2+\mathbb{E}\Big[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\Big]+\mathbb{E}\Big[\int_Q e^{-\alpha s}u_\epsilon\psi_\epsilon(u_\epsilon)dxds\Big]$$
$$+\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big(\frac{\alpha}{2}|u_\epsilon|^2-u_\epsilon w_s(u_\epsilon)-fu_\epsilon-\frac{1}{2}h^2(u_\epsilon)\Big)dxds\Big]=0. \tag{3.2}$$

Now, by passing to the superior limit we get using Remark 3.10

$$\begin{aligned}
&\limsup\Big(e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]+\mathbb{E}\Big[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\Big]\Big)+\mathbb{E}\Big[\int_Q e^{-\alpha s}\xi_2dxds\Big]\\
&+\liminf\Big(\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big(\frac{\alpha}{2}|u_\epsilon|^2-u_\epsilon w_s(u_\epsilon)-\frac{1}{2}h^2(u_\epsilon)\Big)dxds\Big]\Big)\\
\leqslant\ &\mathbb{E}\Big[\int_Q e^{-\alpha s}fudxds\Big]+\frac{1}{2}\|u_0\|^2.
\end{aligned} \tag{3.3}$$

By denoting

$$A_\epsilon=\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big(\frac{\alpha}{2}|u_\epsilon|^2-u_\epsilon w_s(u_\epsilon)-\frac{1}{2}h^2(u_\epsilon)\Big)dxds\Big],$$

we have

$$\begin{aligned}
A_\epsilon\ =\ &\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{\frac{\alpha}{2}|u_\epsilon-u|^2-(u_\epsilon-u)\big(w_s(u_\epsilon)-w_s(u)\big)-\frac{1}{2}\big(h(u_\epsilon)-h(u)\big)^2\Big\}dxds\Big]\\
&+\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{\alpha\big(u_\epsilon u-\frac{u^2}{2}\big)-u_\epsilon w_s(u)-u\big(w_s(u_\epsilon)-w_s(u)\big)-h(u)h(u_\epsilon)+\frac{1}{2}h^2(u)\Big\}dxds\Big]\\
\geqslant\ &\mathbb{E}\Big[\int_Q e^{-\alpha s}|u_\epsilon-u|^2\Big(\frac{\alpha}{2}-L_s-\frac{C_h^2}{2}\Big)dxds\Big]+\mathbb{E}\Big[\int_Q\frac{\alpha}{2}e^{-\alpha s}\big(u_\epsilon u+u(u_\epsilon-u)\big)dxds\Big]\\
&+\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{(u-u_\epsilon)w_s(u)-uw_s(u_\epsilon)-h(u)h(u_\epsilon)+\frac{1}{2}h^2(u)\Big\}dxds\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\liminf A_\epsilon\ \geqslant\ &\liminf\mathbb{E}\Big[\int_Q e^{-\alpha s}|u_\epsilon-u|^2\Big(\frac{\alpha}{2}-L_s-\frac{C_h^2}{2}\Big)dxds\Big]+\mathbb{E}\Big[\int_Q\frac{\alpha}{2}e^{-\alpha s}|u|^2dxds\Big]\\
&+\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{\frac{1}{2}h^2(u)-u\overline{w}_s-h(u)\overline{h}\Big\}dxds\Big].
\end{aligned}$$

Now, by choosing α such that $\frac{\alpha}{2}-L_s-\frac{C_h^2}{2}\geqslant 0$, one gets that

$$\liminf A_\epsilon\ \geqslant\ \mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{\frac{\alpha}{2}|u|^2-u\overline{w}_s+\frac{1}{2}h^2(u)-h(u)\overline{h}\Big\}dxds\Big].$$

Using this in (3.3), we obtain

$$\begin{aligned}
&\limsup\Big(e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]+\mathbb{E}\Big[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\Big]\Big)\\
&+\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{\xi_2+\frac{\alpha}{2}|u|^2-u\overline{w}_s+\frac{1}{2}h^2(u)-h(u)\overline{h}-fu\Big\}dxds\Big]\\
\leqslant\ &\frac{1}{2}\|u_0\|^2.
\end{aligned} \tag{3.4}$$

Besides, by applying the Itô formula to the process u and the function $F(t,v)=\frac{1}{2}e^{-\alpha t}\|v\|^2$, we get, after taking the expectation

$$\begin{aligned}
&e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u(T)\|^2\right]-\frac{1}{2}\|u_0\|^2+\mathbb{E}\Big[\int_Q e^{-\alpha s}|\nabla u|^2dxds\Big]\\
=\ &\mathbb{E}\Big[\int_Q e^{-\alpha s}\Big\{-\frac{\alpha}{2}|u|^2+u\overline{w}_s+\frac{1}{2}\overline{h}^2-\xi u+fu\Big\}dxds\Big].
\end{aligned} \tag{3.5}$$

And by injecting it in (3.4) we get

$$\begin{aligned}
&\limsup\left(e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]+\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\right]\right)\\
&+\mathbb{E}\left[\int_Q e^{-\alpha s}\left\{\xi_2-\xi u+\frac{1}{2}(h(u)-\overline{h})^2\right\}dxds\right]\\
\leqslant\ & e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u(T)\|^2\right]+\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u|^2dxds\right]. \qquad (3.6)
\end{aligned}$$

Thanks to the properties of the weak convergence and of the lim sup and lim inf, one has that

$$\begin{aligned}
&e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u(T)\|^2\right]+\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u|^2dxds\right]\\
&\leq\liminf\left(e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]\right)+\liminf\left(\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\right]\right)\\
&\leq\limsup\left(e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]+\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\right]\right),
\end{aligned}$$

so that

$$\mathbb{E}\left[\int_Q e^{-\alpha s}\left\{\xi_2-\xi u+\frac{1}{2}(h(u)-\overline{h})^2\right\}dxds\right]\leq 0.$$

Thanks to Proposition 3.9 and Remark 3.7, we note that for almost all $(\omega,x,t)\in\Omega\times Q$, $0\leqslant u(\omega,x,t)\leqslant 1$ and $\xi=\xi_1+\xi_2$ with $\xi_1\leq 0$ and $\xi_2\geq 0$.
Then $\xi_2-\xi u=(1-u)\xi_2-\xi_1 u\geq 0$ and a first consequence of the previous inequality is that $h(u)=\overline{h}$. A second consequence is that $\xi_2-\xi u=(1-u)\xi_2-\xi_1 u=0$ and since:

. if $u=0$ then $\xi_2=0$. So $\xi=\xi_1\leqslant 0$ and in this way $\xi\in\mathbb{R}^-$,

. if $u=1$ then $\xi_1=0$. So $\xi=\xi_2\geqslant 0$ and in this way $\xi\in\mathbb{R}^+$,

. if $0<u<1$, then $\xi_1=\xi_2=0$ and $\xi=0$,

one may conclude that $\xi\in\partial I_{[0,1]}(u)$.
The last consequence is that

$$\begin{aligned}
&\limsup\left(e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u_\epsilon(T)\|^2\right]+\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u_\epsilon|^2dxds\right]\right)\\
&=e^{-\alpha T}\frac{1}{2}\mathbb{E}\left[\|u(T)\|^2\right]+\mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla u|^2dxds\right].
\end{aligned}$$

A standard monotone argument yields:

$$u_\epsilon(T)\to u(T)\text{ in }L^2(\Omega\times D),\qquad \nabla u_\epsilon\to\nabla u\text{ in }L^2((0,T)\times\Omega,L^2(D)).$$

Note that since T is arbitrary, one can also conclude the convergence of $u_\epsilon(t)$ to $u(t)$ in $L^2(\Omega\times D)$ for any t. Then, by Proposition 2.2 and the lower semi-continuity of the norm for the weak-* convergence, $\mathbb{E}\left[\|u_\epsilon(t)\|^2\right]$ is bounded uniformly with respect to t and ϵ, Lebesgue converging theorem yields $u_\epsilon\to u$ in $L^2((0,T)\times\Omega,H^1(D))$.
Then, by using the Lipschitz property of w_s one gets that $\overline{w}_s=w_s(u)$ and that the convergence of $w_s(u_\epsilon)$ towards $w_s(u)$ holds strongly in $L^2(\Omega\times Q)$. ■

3.3 Uniqueness of the solution

Theorem 3.12 *Problem (1.1) admits a unique solution.*

Proof. We consider two solutions u and $\hat{u}$ of Problem (1.1) associated with the same initial condition $u(0)=\hat{u}(0)=u_0$:

$$\begin{aligned}
&du+(\xi-\Delta u)dt=(w_s(u)+f)dt+h(u)dW,\quad \xi\in\partial I_{[0,1]}(u)\\
&d\hat{u}+(\hat{\xi}-\Delta\hat{u})dt=(w_s(\hat{u})+f)dt+h(\hat{u})dW,\quad \hat{\xi}\in\partial I_{[0,1]}(\hat{u}).
\end{aligned}$$

Note that, by monotonicity, it holds $(\xi - \hat{\xi})(u - \hat{u}) \geqslant 0$.
Besides, by applying the Itô formula to the process $u - \hat{u}$ and to the function $F(t, v) = e^{-\alpha s}||v||^2$ where $\alpha > 0$, one gets after taking the expectation

$$\frac{1}{2}e^{-\alpha T}\mathbb{E}\left[||(u-\hat{u})(T)||^2\right] + \mathbb{E}\left[\int_Q e^{-\alpha s}|\nabla(u-\hat{u})|^2 dxds\right] + \frac{\alpha}{2}\mathbb{E}\left[\int_Q e^{-\alpha s}|u-\hat{u}|^2 dxds\right]$$
$$\leqslant \quad L_s\mathbb{E}\left[\int_Q e^{-\alpha s}|u-\hat{u}|^2 dxds\right] + \frac{C_h^2}{2}\mathbb{E}\left[\int_Q e^{-\alpha s}|u-\hat{u}|^2 dxds\right].$$

Then, by choosing α such that $\frac{\alpha}{2} - L_s - \frac{C_h^2}{2} \leqslant 0$, one gets that $u = \hat{u}$, and, going back to the equations, one has finally that $\xi = \hat{\xi}$ and the solution is unique. ■

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