Composite Media and Homogenization Theory

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Edited by
Gianni Dal Maso
Gian Fausto Dell'Antonio

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Gianni Dal Maso
Scuola Internazionale Superiore di Studi Avanzati
Trieste, Italy

Gian Fausto Dell'Antonio
Dipartimento di Matematica
Università "La Sapienza"
Roma, Italy

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Contents

HOMOGENIZATION, PLASTICITY AND YIELD DESIGN

G. BOUCHITTE P. SUQUET

Abstract: We consider an epi-convergence problem arising from the theory of yield design. The functional under consideration has a linear growth with respect to the deformation tensor of the displacement field, and the problem is naturally posed in a space of displacement fields with bounded deformation. The problem includes a linear constraint which can be closed or not closed, depending on the type of boundary conditions considered. In the case where the constraint is not closed (applied forces on a part of the boundary) a relaxation term appears. Physically the strength of the loaded boundary turns out to be smaller than the natural guess deduced from the well known Average Variational Principle.

1. OUTLINE OF THE PAPER

This paper deals with an homogenization problem arising from *yield design*, i.e. from the mechanical theory which predicts the load carrying capacity of a structure made from materials with a limited strength. In its dual form, the yield design problem (often called the limit load problem) for a finely periodic structure can be written as a variational problem:

$$(1.1) \qquad \lambda^{\epsilon} = \inf_{\substack{u=0 \text{ sur } \Gamma_0 \\ L(u)=1}} J^{\epsilon}(u) = \int_{\Omega} j^{\epsilon}(x,e(u))\, dx \quad .$$

where Ω is a bounded open set in $\mathbb{R}^N$, $u:\Omega \to \mathbb{R}^N$ is a vector valued field (rate of displacement), $e(u)$ is its deformation tensor, Γ_0 is a subset of the boundary $\partial\Omega$, $L(u)$ is the linear form:

$$(1.2) \qquad L(u) = \int_{\Omega} f_0 . u \, dx + \int_{\Gamma_1} g_0 . u \, ds \quad .$$

where $\Gamma_1 = \partial\Omega - \Gamma_0$.

$j^{\epsilon}(x,E)$ is ϵ-periodic with respect to the variable x, and is positively homogeneous of degree one with respect to the variable E:

$$(1.3) \qquad j^{\epsilon}(x+\epsilon T,E) = j^{\epsilon}(x,E) \text{ for every } T \text{ in } \mathbb{Z}^N,$$

$$(1.4) \qquad j^{\epsilon}(x,\lambda E) = \lambda\, j^{\epsilon}(x,E) \text{ for every } E \text{ in } \mathbb{R}_s^{N^2} \text{ and every } \lambda > 0.$$

Our objective is to find the limit of λ^{ϵ} and of J^{ϵ} (in the sense of Γ-limits) when ϵ goes to 0.

A first guess for this limit, inspired by the *Average Variational Principle* (A.V.P.) (BENSOUSSAN&al [4], MARCELLINI [14]), would be to replace j^{ϵ} in (1.1) by j^{hom} defined by a variational problem on the unit cell $Y =]0,1[^N$, which generates the entire geometry by periodicity:

$$(1.5) \qquad \lambda^0 = \operatorname*{Inf}_{\substack{u=0 \text{ on } \Gamma_0 \\ L(u)=1}} \int_{\Omega} j^{hom}(e(u))\, dx,$$

$$(1.6) \qquad j^{hom}(E) = \operatorname*{Inf}_{w \text{ periodic}} \frac{1}{|Y|} \int_Y j(y,E+e(w))\, dy.$$

This first guess is correct *if $\Gamma_0 = \partial\Omega$, or if g vanishes identically* (i.e. if the boundary is not loaded by imposed external forces). More specifically in this case:

$$(1.7) \qquad \lim_{\epsilon \to 0} \lambda^{\epsilon} = \lambda^0 = \operatorname*{Inf}_{L(u)=1} \int_{\Omega} j^{hom}(e(u))\, dx + \int_{\Gamma_0} j^{hom}(-u \otimes_s n)\, ds.$$

The second integral in (1.7) is a classical relaxation term associated to the first integral term, and accounts for the loss of the boundary condition $u|_{\Gamma_0} = 0$. This term is classical in the theory of minimal surfaces, and in Plasticity TEMAM [17].

Surprisingly, when the boundary is loaded ($\text{mes}(\Gamma_1)>0$, $g\not\equiv 0$) the preceeding guess (1.6) overestimates the exact result, SUQUET[16], and can lead to an uncorrect evaluation of $\lim \lambda^{\epsilon}$. Examples of discrepancies between the guess and the correct limit have been exhibited by DE BUHAN[11] and TURGEMAN&col[17]. A simplified form of these examples is given in the Appendix, together with other considerations on the strength of multilayered materials. The present paper gives a variational formulation of the desired limit (Theorem 2 + corollary 2):

$$(1.8)\ \lim_{\epsilon\to 0} \lambda^\epsilon = \operatorname*{Inf}_{\substack{u,\mu\\ \hat{L}(u,\mu)=1}} \left\{ \int_\Omega j^{hom}(e(u))\,dx + \int_{\Gamma_0} j^{hom}(-u\otimes_s n)\,ds + \int_{\Gamma_1} h(x,\mu-uds) \right\}$$

where : $\hat{L}(u,\mu) = \int_\Omega f_0 . u\,dx + \int_{\Gamma_1} g_0 . \mu$,

and where $h(x,z)$ is a convex function, positively homogeneous of degree one, *which can be strictly smaller than* $j^{hom}(-z\otimes_s n(x))$.

Expressed in mathematical terms, the basic explanation of this result is that the linear form L is not lower semi-continuous in the natural functional space of definition for (1.1). There appears a relaxation term for the constraint $L(u) = 1$, and this relaxation is expressed by μ, $\hat{L}$, and h.

Expressed in physical terms, our results states that *there is a change in the behaviour of the homogenized material on loaded boundaries*. This change in behaviour is better displayed on the primal characterization of λ^ϵ, namely:

$$(1.9)\ \lambda^\epsilon = \operatorname{Sup}\ \{\ \lambda,\ \exists\ \sigma,\ \operatorname{div}(\sigma)+\lambda f_0 = 0,\ \sigma.n = \lambda g_0 \text{ on } \Gamma_0,\ \sigma(x) \in P^\epsilon(x) \text{ a.e. } x \text{ in } \Omega\ \}.$$

where $P^\epsilon(x)$ is the domain of $(j^\epsilon)^*(x,.)$, hereafter called the *strength domain of the material*. The A.V.P. suggests that the limit of λ^ϵ could be:

$$(1.9)\ \lambda^0 = \operatorname{Sup}\ \{\ \lambda,\ \exists\ \sigma,\ \operatorname{div}(\sigma) + \lambda f_0 = 0,\ \sigma.n = \lambda g_0 \text{ on } \Gamma_0,\ \sigma(x) \in P^{hom} \text{ a.e. } x \text{ in } \Omega\ \},$$

where P^{hom} is the domain of $(j^{hom})^*$.

Indeed it is proven in this paper that the limit of λ^ϵ is equal to λ^{hom}:

$$(1.10)\ \lambda^{hom} = \operatorname{Sup}\ \{\ \lambda,\ \exists\ \sigma,\ \operatorname{div}(\sigma)+\lambda f_0 = 0,\ \sigma.n = \lambda g_0 \text{ on } \Gamma_0,\ \sigma(x) \in P^{hom} \text{ a.e. } x \text{ in } \Omega,\ \sigma(x).n(x) \in C(x) \text{ on } \Gamma_1\ \},$$

The convex set $C(x)$, whose detailed derivation will be given in the text, denotes the strength of the material on Γ_1. It can be strictly smaller than the set $C^{hom}(x) = \{\ \sigma.n,\ \sigma \in P^{hom}\}$, indicating that the strength of the homogenized material can be strictly smaller on the loaded boundary than at any interior point of the body.

2. NOTATIONS AND ASSUMPTIONS

2.1. The Mechanical setting

A periodically non homogeneous material with limited strength occupies a domain Ω in $\mathbb{R}^N$. Throughout this paper Ω is supposed to satisfy:

(2.1) Ω is bounded and $\partial\Omega$ is C^1.

For the sake of simplicity it is assumed that $\partial\Omega$ can be shared into two compact disconnected parts Γ_0 and Γ_1.

The condition of limited strength is expressed by the fact that the stress tensor σ (symmetric N×N tensor field) belongs to a *strength domain* $P^\epsilon(x)$ at every point x in Ω. P^ϵ is assumed to be ϵY periodic:

(2.2) $\sigma(x) \in P^\epsilon(x) = P(\frac{x}{\epsilon})$ for every x in Ω.

Throughout the paper we assume that P^ϵ has the following properties:

(2.3) P^ϵ is a closed and convex subset of $\mathbb{R}_s^{N^2}$ (space of symmetric N×N tensors),

(2.4) There exists two strictly positive scalars k_0 and k_1 such that:

$$\{\sigma, |\sigma| \leq k_0\} \subset P^\epsilon(x) \subset \{\sigma, |\sigma| \leq k_1\} \text{ for every x in } \Omega.$$

(2.5) P(y) is constant on smooth subdomains of Y. The typical situation is that of a partition of Y into two subdomains Y_0 and Y_1, called the constituents, and:

$$P(y) = P^1 \text{ if } y \in Y_1, \quad P(y) = P^0 \text{ if } y \in Y_0,$$

where P^1 and P^0 are the strength domains of each constituent.

Ω is loaded by body forces λf_0 and surface forces λg_0 on the part Γ_1 of $\partial\Omega$. λ is the *load parameter*. Equilibrium of the body reads as:

(2.6) $\mathrm{div}(\sigma) + \lambda f_0 = 0$ in Ω, $\sigma.n = \lambda g_0$ on Γ_1.

Throughout the paper we shall assume the following regularity of the loading:

(2.7) $\quad f_0 \in \mathbb{L}^{\infty}(\Omega)\ ,\ g_0 \in \mathbb{C}^0(\Gamma_1)$.

The *limit load*, defined statically (in terms of stress tensors), is:

(2.8) $\lambda^\epsilon = \mathrm{Sup}\ \{\ \lambda \mid$ there exists σ such that:
$$\mathrm{div}(\sigma) + \lambda f_0 = 0 \text{ in } \Omega,\ \sigma.n = \lambda g_0 \text{ on } \Gamma_1,$$
$$\sigma(x) \in P^\epsilon(x) \text{ a.e. } x \text{ in } \Omega \quad \}.$$

λ^ϵ can alternatively be determined by a dual problem TEMAM[17]:

(2.9) $$\lambda^\epsilon = \operatorname*{Inf}_u \left\{ \int_\Omega j^\epsilon(e(u))\ dx\ ;\ u = u_0 \text{ on } \Gamma_0\ ,\ L(u) = 1 \right\},$$

where :

(2.10) $$j^\epsilon(x,.) = (\mathbb{I}_{P^\epsilon(x)})^*\ ,\ L(u) = \int_\Omega f_0.u\ dx + \int_{\Gamma_1} g_0.u\ ds.$$

We shall consider in the sequel a slightly more general version of (2.9), namely:

(2.11) $$\lambda^\epsilon = \operatorname*{Inf}_u \left\{ \int_\Omega j(\frac{x}{\epsilon}, e(u))\ dx\ ;\ u = u_0 \text{ on } \Gamma_0\ ,\ L(u) = 1 \right\},$$

where u_0, and $j(y,E)$ are assumed to obey:

(2.12) $\quad u_0 \in \mathbb{H}^{1/2}(\Gamma_0)$,

$j(y,E)$ is convex and lower semi-continuous on $\mathbb{R}_s^{N^2}$ with respect to E, and periodic with respect to y. Moreover there exists k_0, k_1, k_2 strictly positive constants such that:

(2.13) $\quad k_0\ (|E|-1) \leq j(y,E) \leq k_1\ (|E|+1)$,

(2.14) $\quad j^*(y,\Sigma) \leq k_2$ for every Σ in $P(y) = \mathrm{dom}(j^*(y,.))$.

Note that these assumptions imply in turn the following estimate for the singular part j_∞ of j (defined below):

(2.15) $\quad j_\infty(y,E) \leq j(y,E) + k_2$.

2.2 Functional analysis

We shall extensively use in the sequel the space of vector fields with Bounded Deformation (SUQUET[15], TEMAM[17]):

$$BD(\Omega) = \left\{ u = (u_i)_{i=1,N} \ , \ u_i \in L^1(\Omega) \ , \ \epsilon_{ij}(u) \in M^1(\Omega) \right\}$$

where $M^1(\Omega)$ stands for the space of bounded measures on Ω. Classical results assert that $BD(\Omega)$ has the following properties:

(2.16) * $BD(\Omega)$ is the dual of a Banach space and, therefore, can be endowed with a weak * topology for which bounded sets are relatively compact sets. However *$BD(\Omega)$ is not a reflexive space.*

(2.17) * There exists a trace application from $BD(\Omega)$ onto $L^1(\partial\Omega)^N$, continuous for the strong topologies of these two spaces. However *this trace application is not continuous for the weak * topologies of these spaces.*

The following space will also be useful in the sequel:

(2.18) $$LD(\Omega) = \left\{ u = (u_i)_{i=1,N} \ , \ u_i \in L^1(\Omega) \ , \ \epsilon_{ij}(u) \in L^1(\Omega) \right\}$$

In the following of the paper we shall deal with vector fields or tensorial fields, rather than with scalar fields, and we shall denote by a barred or a curved letter the corresponding functional spaces. For instance:

(2.19) $$\mathbb{L}^p(\Omega) = \left\{ u = (u_i)_{i=1,N} \ , \ u_i \in L^p(\Omega) \right\},$$

(2.20) $$\mathcal{L}^p(\Omega) = \left\{ \sigma = (\sigma_{ij})_{1\leq i,j\leq N} \ , \ \sigma_{ij} = \sigma_{ji} \ , \ \sigma_{ij} \in L^p(\Omega) \right\}$$

2.3. Convex analysis

For a proper convex function $F: X \to \mathbb{R}\cup\{+\infty\}$ (where X is a Banach space) we define its conjugate function $F^*: X^* \to \mathbb{R}\cup\{+\infty\}$ as:

$$F^*(x^*) = \sup_{x \in X} \{ (x,x^*) - F(x) \}$$

F^* is a convex l.s.c. proper function on X^*. Moreover $\bar{F} = (F^*)^*$.

The indicator function of a nonempty, closed and convex set K in X will be denoted by $\mathbb{I}_K$:

$$\mathbb{I}_K(x) = 0 \text{ if } x \in K, \ +\infty \text{ otherwise.}$$

$(\mathbb{I}_K)^*$ is the support function of K. It is a l.s.c., convex function, positively homogeneous of degree one.

We define the recession function j_∞ of a l.s.c., convex function j by:

$$j_\infty(E) = \lim_{t\to+\infty} \frac{1}{t} j\ (tE)\ .$$

It can be easily checked that j_∞ is the support function of dom(j^*).

4. Γ-convergence

We shall frequently refer to the theory of Γ-convergence (DE GIORGI[12], ATTOUCH[2]). Let (X,τ) be a topological vector space, and F^ϵ a sequence of functions mapping X into $\mathbb{R}\cup\{+\infty\}$. We define Γ-liminf($F^\epsilon$) and Γ-limsup($F^\epsilon$) as follows:

$$\text{(2.21)} \qquad \Gamma\text{-liminf}(F^\epsilon)(x) = \text{Inf}\left\{ \text{liminf}(F^\epsilon(x^\epsilon))\ ;\ x^\epsilon \xrightarrow{\tau} x \right\},$$

$$\text{(2.22)} \qquad \Gamma\text{-limsup}(F^\epsilon)(x) = \text{Inf}\left\{ \text{limsup}(F^\epsilon(x^\epsilon))\ ;\ x^\epsilon \xrightarrow{\tau} x \right\}.$$

These two Γ-limits are lower semi-continuous on (X,τ). *F^ϵ is said to be τ- Γ-convergent to F* if:

$$\text{(2.22)} \qquad \Gamma\text{-liminf}(F^\epsilon) = \Gamma\text{-limsup}(F^\epsilon)\ .$$

It is easily checked that F^ϵ is τ-Γ-convergent to F if the two following requirements are met:

i) For every x in X, and every x^ϵ in X τ-converging to x, then

$$\text{(2.23)} \qquad F(x) \le \liminf_\epsilon F^\epsilon(x^\epsilon)$$

ii) For every x in X, there exists x^ϵ in X τ-converging to x such that:

$$\text{(2.24)} \qquad F(x) \ge \limsup_\epsilon F^\epsilon(x^\epsilon)$$

In the sequel we shall omit the "τ" for brevity.

We define the l.s.c. hull of any functional $F: X \to \mathbb{R}\cup\{+\infty\}$ by:

$$\bar{F} = \text{Sup}\{\ G,\ G(x) \le F(x) \text{ for every x in X, G is } \tau \text{ l.s.c.}\}$$

We shall sometimes refer to $\bar{F}$ as the relaxed function associated with F. An interesting property of Γ-convergence is that $\bar{F}$ is the Γ-limit of the sequence $F^\epsilon = F$.

3. HOMOGENIZATION AND RELAXATION

3.1 Preliminary result

Let us first consider the case where $\Gamma_1 = \emptyset$, and define on $\mathbb{L}^p(\Omega)$ (p will be specified later on):

$$(3.1) \qquad J^\epsilon(u) = \begin{cases} \int_\Omega j^\epsilon(e(u))\, dx & \text{if } u \in LD(\Omega), \\ +\infty \text{ otherwise}. \end{cases},$$

$$(3.2) \qquad J^\epsilon_{u_0}(u) = J^\epsilon(u) + \mathbb{I}_{\{u = u_0 \text{ on } \partial\Omega\}}.$$

Let $j^{hom}(E)$ be the homogenized density of energy deduced from the A.V.P.:

$$(3.3) \qquad j^{hom}(E) = \operatorname*{Inf}_{w \in \mathbb{H}^1_{per}} \left\{ \frac{1}{|Y|} \int_Y j(y, E+e(w))\, dy \right\}.$$

It is readily seen that j^{hom} is convex and obeys (2.12) and (2.13). Its conjugate function reads as (BOUCHITTE[5]):

$$(3.4) \qquad (j^{hom})^*(\Sigma) = \operatorname*{Inf}_{\sigma \in S_{per}} \left\{ \frac{1}{|Y|} \int_Y j^*(y, \Sigma+\sigma(y))\, dy \right\},$$

where :

$$S_{per} = \left\{ \sigma \in \mathcal{L}^2(Y),\ \operatorname{div}(\sigma) = 0,\ \int_Y \sigma\, dy = 0,\ \sigma.n \text{ anti-periodic} \right\}.$$

We note $P^{hom} = \operatorname{dom}((j^{hom})^*)$, and we define:

$$(3.5) \qquad J^{hom}(u) = \begin{cases} \int_\Omega j^{hom}(e(u)) & \text{if } u \in BD(\Omega), \\ +\infty \text{ otherwise}. \end{cases}$$

$$(3.6) \qquad J^{hom}_{u_0}(u) = J^{hom}(u) + \int_{\partial\Omega} j^{hom}_\infty((u_0 - u)\otimes_s n)\, ds.$$

THEOREM 1 (BOUCHITTE[5]): Let p be such that $1 \leq p \leq N/(N-1)$. Then:

i) J^{ϵ} Γ-converges to J^{hom} in $\mathbb{L}^{p}(\Omega)$ weak (and strong if $p < N/(N-1)$)

ii) $J^{\epsilon}_{u_0}$ Γ-converges to $J^{hom}_{u_0}$ in $\mathbb{L}^{p}(\Omega)$ weak (and strong if $p < N/(N-1)$.

The reader is referred to [5] for the proof of Theorem 1. This result has been extended to a more general form of j by DEMENGEL & TANG-QI [13].

We now turn to the problem of limit loads. Let L be the linear form:

$$(3.7) \qquad L(u) = \int_{\Omega} f_0 . u \, dx \ , \text{ where f satisfies (2.7),}$$

and consider the variational problems:

$$(3.8) \qquad \lambda^{\epsilon} = \inf_{L(u) = 1} J^{\epsilon}_0(u) \ ,$$

$$(3.9) \qquad \lambda^{hom} = \inf_{L(u) = 1} J^{hom}_0(u) \ .$$

COROLLARY 1: Under the above assumptions:

$$\lim_{\epsilon \to 0} \lambda^{\epsilon} = \lambda^{hom} .$$

Corollary 1 is a direct consequence of Theorem 1, and of the continuity of L on $\mathbb{L}^{p}(\Omega)$. We now turn to the more difficult general case, where a part of the boundary is loaded.

3.2 Loaded boundary: statement of the result

When Γ_1 is not empty, and g_0 does not vanish identically, a new difficulty arise, since the linear form L:

$$(3.10) \qquad L(u) = \int_{\Omega} f_0 . u \, dx + \int_{\Gamma_1} g_0 . u \, ds \ ,$$

is no more continuous on $\mathbb{L}^{p}$ (Ω) or even on $BD(\Omega)$ weak* (the trace operator is not continuous from $BD(\Omega)$ weak* into $\mathbb{L}^{1}(\Gamma_1)$). The constraint $\{ L(u) = 1 \}$ is therefore not closed for the natural topology for which minimizing sequences of the variational problem

(3.8) contain a converging subsequence.

To overcome this difficulty, we consider separately the contribution of the displacement rate u in the inside of Ω and its contribution on the boundary Γ_1 which is denoted by μ. Since μ is in $\mathbb{L}^1(\Gamma_1)$ as soon as u is in $BD(\Omega)$, the most convenient functional space for μ is $\mathbb{M}^1(\Gamma_1)$, where bounded sequences in $\mathbb{L}^1(\Gamma_1)$ contain weakly* convergent subsequences. Let p be such that $1< p < N/(N-1)$ and define on $\mathbb{X}=\mathbb{L}^p(\Omega)\times\mathbb{M}^1(\Gamma_1)$ a sequence of functionals Φ^ϵ by:

$$(3.11)\quad \Phi^\epsilon(u,\mu) = \begin{cases} \int_\Omega j^\epsilon(x,e(u))\,dx & , \text{ if } u \in LD(\Omega)\ ,\quad u|_{\Gamma_0} = u_0\ , \\ & \quad\text{and } \mu = u\,ds \text{ on } \Gamma_1 . \\ +\infty & \text{otherwise} \end{cases}$$

and a linear form, continuous on $\mathbb{X}$, by:

$$(3.12)\qquad \hat{L}(u,\mu) = \int_\Omega f_0 . u\,dx + \int_{\Gamma_1} g_0\,d\mu\ .$$

Note that:

$$(3.13)\quad \lambda^\epsilon = \operatorname*{Inf}_{u,\mu}\ \{\ \Phi^\epsilon(u,\mu)\ ,\quad \hat{L}(u,\mu) = 1\} = \operatorname*{Inf}_{u}\ \{\ J^\epsilon(u)\ ,\ L(u) = 1\}.$$

Since $\hat{L}$ is continuous (while L was not), it is sufficient, in order to pass to the limit in (3.13), to determine the Γ limit of Φ^ϵ on $\mathbb{X}$ endowed with the strong topology of $\mathbb{L}^p(\Omega)$ and the weak* topology of $\mathbb{M}^1(\Gamma_1)$ (note that due to the equi-coercivity of Φ^ϵ, and to the compact embedding of $BD(\Omega)$ into $\mathbb{L}^p(\Omega)$, it would be equivalent to search for the Γ-limit of Φ^ϵ into $BD(\Omega) \times \mathbb{M}^1(\Gamma_1)$ endowed with the weak* topology of each space).

For this purpose, consider:

$$S(\Omega) = \left\{\ \sigma \in \mathcal{L}^\infty(\Omega),\ \operatorname{div}(\sigma) \in \mathbb{L}^{p'}(\Omega)\ ,\ \sigma.n \in \mathbb{C}^0(\Gamma_1)\ \right\},$$

and for f in $\mathbb{L}^{p'}(\Omega)$ and g in $\mathbb{C}^0(\Gamma_1)$:

$$S(f,g) = \{\ \sigma \in S(\Omega),\ \operatorname{div}(\sigma) + f = 0,\ \sigma.n = g \text{ on } \Gamma_1\ \}$$

$S(\Omega)$ is endowed with the following topology τ:

$$(3.14)\quad \sigma^\epsilon \overset{\tau}{\to} \sigma \quad \text{iff} \quad \begin{cases} \sigma^\epsilon \to \sigma \text{ in } \mathcal{L}^\infty(\Omega) \text{ weak*} \\ \operatorname{div}(\sigma^\epsilon) \to \operatorname{div}(\sigma) \text{ in } \mathbb{L}^{p'}(\Omega) \text{ weak *} \\ \sigma^\epsilon.n|_{\Gamma_1} \to \sigma.n|_{\Gamma_1} \text{ in } \mathbb{C}^0(\Gamma_1) \text{ strong} \end{cases}$$

Moreover let us define:

$$\mathbb{K}^\epsilon = \{ \sigma \in S(\Omega),\ \sigma(x) \in P^\epsilon(x) \text{ a.e. } x \text{ in } \Omega \}$$

When ϵ goes to 0, $\mathbb{K}^\epsilon$ admits a limsup, denoted by $\mathbb{K}^s$, and a liminf, denoted by $\mathbb{K}^i$, in Kuratowski's sense for the topology τ. *We further assume that there exists a closed subset* $\mathbb{K}$ *in* $S(\Omega)$ *such that:*

(H) $\mathbb{K}^\epsilon$ *converges to* $\mathbb{K}$ *in Kuratowski's sense for the topology* τ *on* $S(\Omega)$.

Then we define for every x in Γ_1:

$$C(x) = \text{Closure}\{ \sigma.n(x),\ \sigma \in \mathbb{K} \}.$$

C(x) is a l.s.c. multi-application with convex and closed values.

We are now in a position to state our main result:

<u>*THEOREM 2:*</u> *The* Γ*-limit of* Φ^ϵ *in* $\mathbb{X}$ *is:*

$$\Phi^{hom}(u,\mu) = \int_\Omega j^{hom}(e(u)) + \int_{\Gamma_0} j^{hom}_\infty((u_0-u)\otimes_s n)\, ds + \int_{\Gamma_1} h(x,\mu-uds),$$

where $h(x,z)$ *is the l.s.c., convex, positively homogeneous of degree 1 function defined as:*

$$h(x,z) = \mathbb{I}^*_{C(x)}(z).$$

Consider now the sequence λ^ϵ of infima (3.13) (where $u_0 = 0$), and set:

$$(3.15) \qquad \lambda^{hom} = \operatorname*{Inf}_{u,\mu \in \mathbb{X}} \{ \Phi^{hom}(u,\mu)\ ,\ \hat{L}(u,\mu) = 1 \}.$$

<u>*COROLLARY 2:*</u> *Under the above assumptions*:

$$\lim_{\epsilon\to 0} (\lambda^\epsilon) = \lambda^{hom}.$$

Corollary 2 is a direct consequence of Theorem 2 and of the continuity of $\hat{L}$ on X.

Remark 1: It can be checked (cf [7]) that:

$$\lambda^{hom} = \mathrm{Inf}(\ \lambda^0\ ,\ \Lambda\ ),$$

where:

$$\lambda^0 = \mathop{\mathrm{Inf}}_{L(u)\,=\,1} \left\{ \ J_0^{hom}(u) \ \right\}, \text{ and}$$

$$\Lambda = \mathrm{Inf} \left\{ \int_{\Gamma_1} h(x,\mu)\ ,\ \int_{\Gamma_1} g\mu = 1\ ,\ \mu \in \mathbb{M}^1(\Gamma_1) \right\}$$

$$= \mathrm{Sup}\ \{\ \lambda\ ,\ \lambda g(x) \in C(x) \text{ for every } x \text{ in } \Gamma_1\ \}$$

We begin the proof of Theorem 2 with a preliminary result.

PROPOSITION 1: Let σ^ϵ be a sequence of elements of $S(\Omega)$ converging to σ in $S(\Omega)$ for the above described topology τ. Then:

$$\text{(3.16)} \qquad \liminf_{\epsilon\to 0} \int_\Omega (j^\epsilon)^*(\sigma^\epsilon)\ dx \geq \int_\Omega (j^{hom})^*(\sigma)\ dx \quad .$$

Moreover if σ^ϵ belongs to $\mathbb{K}^\epsilon$, then σ belongs to $\mathbb{K}^{hom}$.

Proof of Proposition 1:

Let $(\Omega_i\ )_{i\in I}$ be a finite family of disconnected, Lipschitzian open subsets of Ω, such that

$$\mathrm{mes}(\Omega - \bigcup_{i\in I} \Omega_i\ ) = 0.$$

Let:

$$\text{(3.17)} \qquad z = \sum_{i\in I} z_i\ \chi_{\Omega_i}(x)\ ,$$

where $z_i \in \mathbb{R}_s^{N^2}$, and χ_A is the characteristic function of A. Application of Theorem 1 to each open set Ω_i and to the sequence of functions $j^\epsilon(x,z_i+.)$, yields the existence of a sequence u_ϵ^i in $BD(\Omega_i)$ such that:

i) $u^i_\epsilon \,|_{\partial\Omega_i} = 0$, $\lim_{\epsilon\to 0}(u^i_\epsilon) = 0$ in $\mathbb{L}^{p'}(\Omega_i)$,

ii) $$\lim_{\epsilon\to 0}\int_{\Omega_i} j^\epsilon(z_i + e(u^i_\epsilon))\,dx = \int_{\Omega_i} j^{hom}(z_i)\,dx.$$

Let $u^\epsilon = \sum_{i\in I} u^i_\epsilon\, \chi_{\Omega_i}$. Then:

i) $u^\epsilon|_{\partial\Omega} = 0$, $\lim_{\epsilon\to 0}(u^\epsilon) = 0$ in $\mathbb{L}^{p'}(\Omega)$,

ii) $$\lim_{\epsilon\to 0}\int_{\Omega} j^\epsilon(Z+e(u^\epsilon))\,dx = \int_{\Omega} j^{hom}(Z)\,dx.$$

Now let σ^ϵ be a sequence in $S(\Omega)$ τ-converging to σ. Fenchel's inequality yields:

$$\int_\Omega (j^\epsilon)^*(\sigma^\epsilon)\,dx \geq \int_\Omega \sigma^\epsilon:(Z+e(u^\epsilon))\,dx - \int_\Omega j^\epsilon(Z+e(u^\epsilon))\,dx.$$

But:

$$\lim_{\epsilon\to 0}\int_\Omega \sigma^\epsilon:e(u^\epsilon)\,dx = -\lim_{\epsilon\to 0}\int_\Omega \operatorname{div}(\sigma^\epsilon).u^\epsilon\,dx = 0,$$

therefore:

$$\text{(3.18)}\qquad \liminf_{\epsilon\to 0}\int_\Omega (j^\epsilon)^*(\sigma^\epsilon)\,dx \geq \int_\Omega (\sigma:Z - j^{hom}(Z))\,dx.$$

(3.18) is valid for every piecewise constant Z in the form (3.17), and can be extended by density to every Z in $\mathcal{L}^1(\Omega)$. We take the supremum of the right hand side of (3.18) on Z in $\mathcal{L}^1(\Omega)$, and note that by Rockafellar's theorem:

$$\text{(3.19)}\qquad \operatorname*{Sup}_{Z\in\mathcal{L}^1(\Omega)}\left(\int_\Omega (\sigma:Z - j^{hom}(Z))\,dx\right) = \int_\Omega (j^{hom})^*(\sigma)\,dx\,.$$

(3.16) is now a direct consequence of (3.18) and (3.19).

If moreover σ^ϵ belongs to $\mathbb{K}^\epsilon$, then the left hand side of (3.16) is finite. The same conclusion holds for the right hand side, and $\sigma(x)$ is for a.e. x in the domain of $(j^{hom})^*$, i.e. in P^{hom}. Q.E.D.

<u>Remark 2:</u> Proposition 1 can be strengthened in the following way . Let:

$$G^\epsilon(\sigma) = \int_\Omega (j^\epsilon)^*(\sigma)\ dx\ ,\quad G^{hom}(\sigma) = \int_\Omega (j^{hom})^*(\sigma)\ dx.$$

Then G^ϵ Γ-converges to G^{hom} in $S(\Omega)$ endowed with the topology τ. This stronger result will not be useful in the sequel.

3.3 Proof of Theorem 2. First step: liminf Φ^ϵ

In a first step we show that:

$$\liminf \Phi^\epsilon \geq \Phi^{hom}\ . \tag{3.20}$$

For the sake of simplicity we only consider the case $\Gamma_1 = \partial\Omega$.
Let $(u^\epsilon,\mu^\epsilon)$ be a sequence in X converging to (u,μ) for the topology $\mathbb{L}^p(\Omega)$ strong $\times$ $\mathbb{M}^1(\Gamma_1)$ weak $*$, and such that:

i) $\Phi^\epsilon(u^\epsilon,\mu^\epsilon) \leq C$,
(which implies that $u^\epsilon \in LD(\Omega)$ and $\mu^\epsilon = u^\epsilon ds$)

ii) $\lim_{\epsilon\to 0} (u^\epsilon) = u$ in $\mathbb{L}^p(\Omega)$ strong,
$\lim_{\epsilon\to 0} (u^\epsilon ds) = \mu$ in $\mathbb{M}^1(\partial\Omega)^N$ weak*,

Let σ be an element of $\mathbb{K}$, and σ^ϵ a sequence of elements of $\mathbb{K}^\epsilon$ τ-converging to σ. Consider, after ANZELLOTTI [1], the measure $\lambda^\epsilon = \sigma^\epsilon:e(u^\epsilon)$ defined on Ω by:

$$\langle\lambda^\epsilon,\varphi\rangle = -\int_\Omega \operatorname{div}(\sigma^\epsilon).u^\epsilon\varphi\ dx - \int_\Omega \sigma^\epsilon:(u^\epsilon\otimes\operatorname{grad}(\varphi))\ dx. \tag{3.21}$$

The topologies for which σ^ϵ and u^ϵ are converging sequences allow us to pass to the limit in the right hand side of (3.21):

$$\lim_{\epsilon\to 0} \langle\lambda^\epsilon,\varphi\rangle = \langle\lambda,\varphi\rangle = -\int_\Omega \operatorname{div}(\sigma).u\varphi\ dx - \int_\Omega \sigma:(u\otimes\operatorname{grad}(\varphi))\ dx.$$

Therefore λ^ϵ converges in $M^1(\Omega)$ weak* to $\lambda = \sigma:e(u)$. Define:

$$\Omega_\alpha = \{\ x \in \Omega,\ \operatorname{dist}(x,\partial\Omega) > \alpha\ \}$$

Then a classical argument asserts that, for α outside a countable set, the following convergence holds:

$$\lim_{\epsilon\to 0} \lambda^\epsilon(\Omega_\alpha) = \lambda(\Omega_\alpha)\ . \tag{3.22}$$

Consider such an α as fixed for a moment. Then:

$$(3.23)\quad \int_{\Omega} j^{\epsilon}(e(u^{\epsilon}))\,dx = \int_{\Omega_\alpha} j^{\epsilon}(e(u^{\epsilon}))\,dx + \int_{\Omega-\Omega_\alpha} j^{\epsilon}(e(u^{\epsilon}))\,dx,$$

and by Theorem 1:

$$(3.24)\quad \lim_{\epsilon\to 0} \int_{\Omega_\alpha} j^{\epsilon}(e(u^{\epsilon}))\,dx \geq \int_{\Omega_\alpha} j^{hom}(e(u))\,dx.$$

Next we note that:

$$\int_{\Omega-\Omega_\alpha} j^{\epsilon}(e(u^{\epsilon}))\,dx \geq \int_{\Omega-\Omega_\alpha} j^{\epsilon}_{\infty}(e(u^{\epsilon}))\,dx - k_2\,|\Omega-\Omega_\alpha|\ ,$$

$$(3.25)\qquad \geq \int_{\Omega-\Omega_\alpha} \sigma^{\epsilon}:e(u^{\epsilon})\,dx - k_2\,|\Omega-\Omega_\alpha|\ .$$

But:

$$\int_{\Omega} \sigma^{\epsilon}:e(u^{\epsilon})\,dx = \int_{\partial\Omega} \sigma^{\epsilon}.n.u^{\epsilon}\,ds - \int_{\Omega} \mathrm{div}(\sigma^{\epsilon}).u^{\epsilon}\,dx.$$

Considering the convergence of each term under the integrals in the right hand side of this equality, we obtain on one hand:

$$\lim_{\epsilon\to 0} \int_{\Omega} \sigma^{\epsilon}:e(u^{\epsilon})\,dx = \int_{\partial\Omega} \sigma.n\,d\mu - \int_{\Omega} \mathrm{div}(\sigma).u\,dx\ .$$

On the other hand, by virtue of (3.22), we have:

$$\lim_{\epsilon\to 0} \int_{\Omega_\alpha} \sigma^{\epsilon}:e(u^{\epsilon})\,dx = \int_{\Omega_\alpha} \sigma:e(u)\,dx\ .$$

Therefore:

$$\lim_{\epsilon\to 0} \int_{\Omega-\Omega_\alpha} \sigma^{\epsilon}:e(u^{\epsilon})\,dx = \int_{\partial\Omega} \sigma.n\,d\mu - \int_{\Omega} \mathrm{div}(\sigma).u\,dx - \int_{\Omega_\alpha} \sigma:e(u)\,dx$$

Coming back to (3.23)(3.24) and (3.25) we obtain:

$$\liminf_{\epsilon\to 0} \Phi^{\epsilon}(u^{\epsilon},\mu^{\epsilon}) \geq \int_{\Omega_\alpha} j^{hom}(e(u)) + \int_{\partial\Omega} \sigma.n\,d\mu - \int_{\Omega} \mathrm{div}(\sigma).u\,dx$$
$$-\int_{\Omega_\alpha} \sigma:e(u)\,dx - k_2\,|\Omega-\Omega_\alpha|\ .$$

Now letting α go to 0, and after due use of Green's formula, we obtain:

$$(3.26)\qquad \liminf_{\epsilon\to 0} \Phi^{\epsilon}(u^{\epsilon},\mu^{\epsilon}) \geq \int_{\Omega} j^{hom}(e(u)) + \int_{\partial\Omega} \sigma.n\ (d\mu - uds).$$

(3.26) is valid for every σ in $\mathbb{K}$, and we take the supremum of its right hand side with respect to such σ's. To compute this supremum we claim that:

$$(3.27)\qquad \sup_{\sigma\in\mathbb{K}} \int_{\partial\Omega} \sigma.n.(d\mu - uds) = \int_{\partial\Omega} \sup_{z\in C(x)} (z.(d\mu - uds))$$

$$= \int_{\partial\Omega} h(x,\mu - uds).$$

To prove this claim it is sufficient to prove that the set $\{ \sigma.n,\ \sigma \in \mathbb{K} \}$ is stable under Lipschitzian partition of unity (see BOUCHITTE & VALADIER [8]). This is the object of Lemma 1 below. (3.27) (3.26) complete the proof of (3.20).

<u>*LEMMA 1*</u>*: Define* $\mathbb{K}_1 = \{ \sigma.n,\ \sigma \in \mathbb{K} \}$. *Let* $(\varphi_i)_{i\in I}$ *be a finite family of elements in* $\mathbb{K}_1$, *and* $(\alpha_i)_{i\in I}$ *a Lipschitzian partition of unity on* $\partial\Omega$:

$$\alpha_i \in Lip(\partial\Omega,[0,1])\ ,\ \sum_{i\in I} \alpha_i = 1.$$

Then: $$\varphi = \sum_{i\in I} \alpha_i\ \varphi_i \text{ belongs to } \mathbb{K}_1 .$$

<u>Proof of Lemma 1:</u>

By definition of $\mathbb{K}_1$ there exists a family $(\sigma_i)_{i\in I}$ of elements in $\mathbb{K}$ such that $\varphi_i = \sigma_i.n$ on $\partial\Omega$. According to BOUCHITTE & VALADIER [8], the partition of unity $(\alpha_i)_{i\in I}$ on $\partial\Omega$ can be extended into a Lipschitzian partition of unity $(\beta_i)_{i\in I}$ on $\bar{\Omega}$.

Then $\sigma = \sum_{i\in I} \beta_i\ \sigma_i$ satisfies :

$$\varphi = \sigma.n \text{ on } \partial\Omega.$$

To complete the proof of Lemma 1 it remains to prove that σ belongs to $\mathbb{K}$. By definition of $\mathbb{K}$, each σ_i can be approached in the topology (3.14) by a sequence σ_i^{ϵ} of elements in $\mathbb{K}^{\epsilon}$. Define $\sigma^{\epsilon} = \sum_{i\in I} \beta_i\ \sigma_i^{\epsilon}$. It is easily checked that σ^{ϵ} belongs to $\mathbb{K}^{\epsilon}$ (convexity of P^{ϵ}, and Lipschitz regularity of β_i), and moreover that σ^{ϵ} converges to σ in the topology (3.14). Therefore σ belongs to $\mathbb{K}$, and Lemma 1 is proved. Q.E.D

3.4 Proof of Theorem 2. Second step: limsup Φ^{ϵ}

In a second step we show that:

$$(3.30) \qquad \Gamma\text{-}\limsup_{\epsilon\to 0} \Phi^{\epsilon} \leq \Phi^{hom}.$$

LEMMA 2: In the duality between $\mathbb{X}$ *and* $\mathbb{L}^{p'}(\Omega)\times\mathbb{C}^0(\Gamma_1)$, *the conjugate functions of* Φ^{ϵ} *and* Φ^{hom} *read as:*

$$(3.31) \quad (\Phi^{\epsilon})^*(f,\varphi) = \operatorname*{Inf}_{\sigma\in S(f,\varphi)} \left\{ \int_{\Omega} (j^{\epsilon})^*(\sigma)\, dx - \int_{\Gamma_0} \sigma.n.u_0\, ds \right\}$$

$$(3.32) \quad (\Phi^{hom})^*(f,\varphi) = \begin{cases} \displaystyle\operatorname*{Sup}_{\sigma\in S(f,\varphi)} \int_{\Omega} (j^{hom})^*(\sigma)\, dx - \int_{\Gamma_1} \sigma.n.u_0\, ds \\ \quad \text{if } \varphi(x) \in C(x) \text{ for every } x \text{ in } \Gamma_1, \\ \quad +\infty \text{ otherwise} \end{cases}$$

Let us first prove how (3.30) can be deduced from Lemma 2. For this purpose it is sufficient (AZE [3]) to establish the following inequality:

$$(3.33) \qquad \Gamma\text{-}\liminf(\Phi^{\epsilon})^* \geq (\Phi^{hom})^*.$$

Let $(f^{\epsilon}, \varphi^{\epsilon})$ be a converging sequence in $\mathbb{L}^{p'}(\Omega)$ weak$\times\mathbb{C}^0(\Gamma_1)$ strong, with limit (f,φ), and such that :

$$\liminf\, (\Phi^{\epsilon})^*(f^{\epsilon},\varphi^{\epsilon}) < +\infty .$$

According to Lemma 2, there exists a sequence σ^{ϵ} in $S(\Omega)$ such that:

i) $\sigma^{\epsilon} \in S(f^{\epsilon},\varphi^{\epsilon})$, thus $\sigma^{\epsilon} \in \mathbb{K}^{\epsilon}$,

$$(3.34) \quad \text{ii)}\ (\Phi^{\epsilon})^*(f^{\epsilon},\varphi^{\epsilon}) \geq \int_{\Omega} (j^{\epsilon})^*(\sigma^{\epsilon})\, dx - \int_{\Gamma_0} \sigma^{\epsilon}.n.u_0\, ds - \epsilon .$$

It can be readily seen that the growth condition (2.13) together with (3.34) implies that σ^{ϵ} is bounded in $\mathcal{L}^{\infty}(\Omega)$. Since it belongs to $S(f^{\epsilon},\varphi^{\epsilon}) \cap \mathbb{K}^{\epsilon}$ it contains a τ-converging subsequence, the limit of which, noted σ, is in $S(f,\varphi) \cap \mathbb{K}$. Moreover :

(3.35) $$\lim_{\epsilon\to 0} \int_{\Gamma_0} \sigma^\epsilon.n.u_0 \, ds = \int_{\Gamma_0} \sigma.n.u_0 \, ds \, .$$

According to Proposition 1:

(3.36) $$\liminf_{\epsilon\to 0} \int_\Omega (j^\epsilon)^*(\sigma^\epsilon) \, dx \geq \int_\Omega (j^{hom})^*(\sigma) \, dx \, .$$

Coming back to (3.34) with the help of (3.35)(3.36), we obtain:

(3.37) $$\liminf_{\epsilon\to 0} (\Phi^\epsilon)^*(f^\epsilon,\varphi^\epsilon) \geq (\Phi^{hom})^*(f,\varphi),$$

which is exactly the desired statement (3.30). The proof of Theorem 2 is complete, provided that we prove Lemma 2.

Proof of Lemma 2:

For (f,φ) in $\mathbb{L}^{p'}(\Omega)\times\mathbb{C}^0(\Gamma_1)$ we compute:

$$(\Phi^\epsilon)^*(f,\varphi) = \operatorname*{Sup}_{u,\mu \in X} \left(\int_\Omega f.u \, dx + \int_{\Gamma_1} \varphi.d\mu - \Phi^\epsilon(u,\mu) \right)$$

(3.38) $$= - \operatorname*{Inf}_{u} \left(\int_\Omega j^\epsilon(e(u)) \, dx - \int_\Omega f.u \, dx - \int_{\Gamma_1} \varphi.u \, ds \right)$$

where in (3.38) $u \in LD(\Omega)$ and $u = u_0$ on Γ_0. The computation of the dual problem associated with (3.38) is a routine exercise in Convex Analysis (see TEMAM[17]), and yields (3.31).

The derivation of (3.22) contains a difficulty, since Φ^{hom} is a priori defined on the non reflexive space $BD(\Omega)\times\mathbb{M}^1(\Gamma_1)$, for which application of Convex Analysis is not straightforward. Therefore we consider in a first step the following function:

(3.39) $$\Psi(u,\mu) = \begin{cases} \Phi^{hom}(u,\mu) \text{ if } u \in LD(\Omega) \, , \, \mu \in \mathbb{L}^1(\Gamma_1), \\ +\infty \text{ otherwise.} \end{cases}$$

We claim that Ψ and Φ^{hom} have the same dual functions, or in other words that they have same l.s.c. regularized functions in $\mathbb{L}^p(\Omega)\times \mathbb{M}^1(\Gamma_1)$. The following set of inequalities is straightforward:

$$\Psi \geq \Phi^{hom} \Rightarrow \bar{\Psi} \geq \bar{\Phi}^{hom}.$$

In order to prove the reverse inequality ($\bar{\Psi} \leq \bar{\Phi}^{hom}$), consider, for every (u,μ) in $BD(\Omega)\times\mathbb{M}^1(\Gamma_1)$, a sequence $(u^\epsilon, \theta^\epsilon)$ in $LD(\Omega)\times\mathbb{L}^1(\Omega)$ such that:

i) $\lim_{\epsilon\to 0} u^\epsilon = u$ in $\mathbb{L}^p(\Omega)$, $u^\epsilon = u$ on $\partial\Omega$, and

$$\lim_{\epsilon\to 0} \int_\Omega j^{hom}(e(u^\epsilon))\, dx = \int_\Omega j^{hom}(e(u)) .$$

Existence of such a sequence is ensured by TEMAM[17] (see also DAL MASO [10]).

ii) $\lim_{\epsilon\to 0} \theta^\epsilon = \mu$ in $\mathbb{M}^1(\Gamma_1)$ weak*, and

$$\lim_{\epsilon\to 0} \int_{\Gamma_1} h(x,\theta^\epsilon - u)\, ds = \int_{\Gamma_1} h(x,\mu-uds).$$

Existence of such a sequence results from BOUCHITTE & VALADIER [9] since $h(x,.) = \mathbb{I}^*_{C(x)}(.)$ where C is a l.s.c. multi-application with closed convex values. For this sequence we obtain:

$$\lim_{\epsilon\to 0} \Psi(u^\epsilon,\theta^\epsilon) = \Phi^{hom}(u,\mu) ,$$

i.e. $\bar\Psi \le \Phi^{hom}$ and consequently $\bar\Psi \le \bar\Phi^{hom}$. This inequality completes the proof of the fact that Ψ and Φ^{hom} have the same dual function.

We now proceed to the computation of Ψ^*. For (f,φ) in $\mathbb{L}^{p'}(\Omega)\times \mathbb{C}^0(\Gamma_1)$, $\Psi^*(f,\varphi)$ is defined as:

$$\Psi^*(f,\varphi) = \operatorname*{Sup}_{u,\theta} \left\{ \int_\Omega (f.u - j^{hom}(e(u)))\, dx + \int_{\Gamma_1} (\theta.\varphi - h(x,\theta-u))\, ds \right\}$$

where the Sup is taken over $(u,\theta) \in LD(\Omega)\times\mathbb{L}^1(\Gamma_1)$, $u = u_0$ on Γ_0. For fixed u the Supremum in θ is computed by means of Rockafellar's theorem:

$$\operatorname*{Sup}_{\theta\in\mathbb{L}^1(\Gamma_1)} \left\{ \int_{\Gamma_1} (\theta.\varphi - h(x,\theta-u))\, ds \right\} = \int_{\Gamma_1} (\theta.u + h^*(x,\varphi(x)))\, ds =$$

$$= \begin{cases} \int_{\Gamma_1} \theta.u\, ds \text{ if } \varphi(x) \in C(x) \text{ for every } x \text{ in } \Gamma_1, \\ +\infty \text{ otherwise} \end{cases}$$

Therefore:

$$(3.40)\quad \Psi^*(f,\varphi) = \begin{cases} -\operatorname*{Inf}_{u} \left\{ \int_\Omega (j^{hom}(e(u)) - f.u)\, dx - \int_{\Gamma_1} \varphi.u\, ds, \right\} \\ \qquad \text{if } \varphi(x) \in C(x) \text{ for every } x \text{ in } \Gamma_1, \\ \qquad +\infty \text{ otherwise.} \end{cases}$$

where the infimum in (3.40) is taken over $u \in LD(\Omega)$, $u = u_0$ on Γ_0. We can now perform a routine computation using Convex Analysis, to evaluate the infimum in (3.40) as the supremum in (3.32). Q.E.D

3.5 Comments about the assumption (H)

For practical use, we need to check assumption (H), and to determine more explicitely the convex set $C(x)$ or its support function $h(x,z)$: this is a difficult problem. However the determination of another set $\hat{C}(x)$ gives an useful estimate on $C(x)$. In several cases, the limit in Kuratowski's sense in $\mathbb{C}^0(\Gamma_1)$ strong of the sets :

$$\mathbb{K}_1^\epsilon = \left\{ \sigma.n|_{\Gamma_1},\ \sigma \in \mathbb{K}^\epsilon \right\},$$

can be more easily determined. Indeed it can been proved (BOUCHITTE [6]) that, if $\mathbb{K}_1^\epsilon$ converges in Kuratowski's sense in $\mathbb{C}^0(\Gamma_1)$ to $\mathbb{K}_1$, then $\mathbb{K}_1$ reads as:

$$\mathbb{K}_1 = \left\{ \varphi \in C^0(\Gamma_1)^N,\ \varphi(x) \in \hat{C}(x) \right\},$$

where $\hat{C}$ is a l.s.c. multi-application, with closed convex values.

In the case of two constituents, with strength domains P_0 and P_1, an explicit formula for $\hat{C}(x)$ can be derived. Specifically, let us define:

$$\Omega_1^\epsilon = \left\{ x \in \Omega,\ P^\epsilon(x) = P_1 \right\},\ A^\epsilon = \Gamma_1 \cap \bar{\Omega}_1^\epsilon .$$

We assume that:

$$(3.41)\quad H^{N-1}(\partial A^\epsilon) = 0\ ,\quad Int(A^\epsilon) \to A\ ,\ Int(\Gamma_1 - A^\epsilon) \to B\ .$$

Then ([6]) A and B are closed and:

$$(3.42)\qquad \hat{C}(x) = \begin{cases} P_1\ n(x) \text{ if } x \in A-B\ , \\ P_0\ n(x) \text{ if } x \in B-A\ , \\ P_1\ n(x) \cap P_0\ n(x) \text{ if } x \in A \cap B. \end{cases}$$

More generally let us define:

$$\mathbb{K}^{hom} = \{ \sigma \in S(\Omega) \ , \ \sigma(x) \in P^{hom} \text{ a.e. } x \text{ in } \Omega \} \ ,$$

$$C^{hom}(x) = \text{closure } \{ \sigma.n(x), \ \sigma \in \mathbb{K}^{hom} \} \ .$$

Under the hypothesis (2.1) assuming that $\partial\Omega$ is C^1, it can be checked that $C^{hom}(x) = P^{hom}.n(x)$ (see lemma 3 below). The following proposition relates $C(x)$, $\hat{C}(x)$ and $C^{hom}(x)$.

<u>*PROPOSITION 2*</u>: *Under the above assumptions:*

i) $(P_1 \cap P_0)\, n(x) \subset C(x) \subset \hat{C}(x) \cap C^{hom}(x)$,

ii) $h(x,z) \leq j_\infty^{hom}(n(x) \otimes_s z)$.

The inclusion in i) can be strict as well as the inequality in ii).

<u>Remark 3</u>: Note that a more specific result can be established if:

(3.43) $$P_0 \subset P_1 .$$

Indeed, in this case :

$$(P_1 \cap P_0) n(x) = P_0 n(x) = P_1 n(x) \cap P_0 n(x) \ ,$$

and we deduce from (3.42) and Proposition 2 that:

(3.44) $$C(x) = P_0 n(x) \text{ for every } x \text{ in } B.$$

On B the strength of the boundary is ruled by the strength of the weakest material. It is proven in the appendix that, in case of a layered composite material satisfying (3.43), this conclusion holds true for the entire boundary.

<u>Proof of Proposition 2</u> :

We begin with a preliminary result:

<u>*LEMMA 3*</u>: *Under assumption (2.1) the following equality holds:*

$$C^{hom}(x) = P^{hom}.n(x) \text{ for every } x \text{ in } \Gamma_1 .$$

<u>Proof of Lemma 3</u>: First note that the inclusion

$$P^{hom}.n(x) \subset C^{hom}(x)$$

is immediate: if Z is in $P^{hom}.n(x)$ for one x in Γ_1, there exists Σ in P^{hom} such that $Z = \Sigma.n(x)$. The constant field $\sigma(x) = \Sigma$ is obviously in $\mathbb{K}^{hom}$, and by definition of $C^{hom}(x)$, $\Sigma.n(x)$ belongs to it. In order to prove the reverse inclusion, we note, after ANZELLOTTI, that for σ in $\mathcal{L}^{\infty}(\Omega)$ and $\operatorname{div}(\sigma)$ in $\mathbb{L}^{p'}(\Omega)$, and provided that $\partial\Omega$ is C^1, we have the following characterization of $\sigma.n$ on $\partial\Omega$: at every Lebesgue point of $\sigma.n$

$$\sigma.n(x) = \lim_{\rho\to 0^+} \lim_{r\to 0^+} \left\{ \frac{1}{|Q_{r,\rho}(x)|} \int_{Q_{r,\rho}(x)} \sigma.n(y)\, dy \right\}$$

where $Q_{r,\rho}(x) = \{ y - tn(x),\ |y-x| < \rho,\ 0 < t < r \}$. Therefore $\sigma.n(x)$ belongs to $P^{hom}.n(x)$ a.e. x in $\partial\Omega$, as soon as $\sigma(y)$ belongs to P^{hom} a.e. y in Ω. If moreover σ belongs to $\mathbb{K}^{hom}$, the continuity of both $\sigma.n(x)$ and of the multi-application $P^{hom}.n(x)$ imply that $\sigma.n(x)$ is in $P^{hom}.n(x)$ for every x in Γ_1. Therefore $P^{hom}.n(x)$ contains $\{ \sigma.n(x) ,\ \sigma \in \mathbb{K}^{hom}\}$, and since it is closed, it contains the closure of this set, which is exactly $C^{hom}(x)$. Q.E.D.

Coming back to the proof of proposition 2, we note that the first inclusion in i) is straightforward. Let us prove the second one. Let Z be an element of $C(x)$, and $Z' = \lambda Z$ with $0 < \lambda < 1$. There exists φ in $\mathbb{C}^0(\Gamma_1)$ such that $\varphi(x) = Z'$, and σ in $\mathbb{K}$ such that $\sigma.n = \varphi$ on Γ_1. Let σ^{ϵ} be a sequence of elements in $\mathbb{K}^{\epsilon}$ converging to σ for the topology (3.14). Then, according to Proposition 1:

$$\sigma(x) \in P^{hom} \text{ a.e. } x \text{ in } \Omega,$$

and therefore $\varphi(x) = \sigma.n(x) \in P^{hom}n(x)$ for every x in Γ_1. Moreover:

$$\lim_{\epsilon\to 0} \sigma^{\epsilon}.n|_{\Gamma_1} = \sigma.n|_{\Gamma_1} \text{ in } \mathbb{C}^0(\Gamma_1)\ ,$$

and therefore $\varphi(x) = \sigma.n(x) \in \hat{C}(x)$ for every x in Γ_1, and more specifically that Z' belongs to $\hat{C}(x)$. The conclusion is extended to Z by letting λ tend to 1. This completes the proof of the second inclusion in i). ii) is a direct consequence of this second inclusion.

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APPENDIX
Layered materials

A.1 Strength domain of a layered material with two constituents

Consider a two dimensional layered medium with two constituents denoted 0 and 1, with volume fractions v_0 and v_1. The strength domain of each constituent is defined by :

(A.1) $$\sigma_{11}^2 + 2\sigma_{12}^2 + \sigma_{22}^2 \leq k^2(x) ,$$

where $k(x) = k_0$ or k_1, and $k_0 \leq k_1$.

The two layers being infinite in the x_1 direction, the microscopic stress fields can be assumed to depend only on x_2. Therefore it results from the equilibrium equations that σ_{12} and σ_{22} are constant and equal to their average:

(A.2) $$\sigma_{12} = \Sigma_{12} , \sigma_{22} = \Sigma_{22} .$$

The microscopic yield condition (A.1) now reads as:

$$|\sigma_{11}(x_2)| \leq \left(k^2(x) - 2\Sigma_{12}^2 + \Sigma_{22}^2\right)^{1/2},$$

i.e.

$$|\Sigma_{11}| \leq v_0 \left(k_0^2 - 2\Sigma_{12}^2 + \Sigma_{22}^2\right)^{1/2} + v_1 \left(k_1^2 - 2\Sigma_{12}^2 + \Sigma_{22}^2\right)^{1/2}.$$

For biaxial stress states ($\Sigma_{12} = 0$, Σ_{11} and $\Sigma_{22} \neq 0$) the above macroscopic strength domain is delimited by fourth order curves, and by the two straight lines $\Sigma_{22} = \pm k_0$ (see figure A.1). There is a *"weakest link" effect* since the strength in the direction orthogonal to the layers is always equal to the smaller strength of the two materials, irrespective of the volume fraction of the constituents.

However in the direction of the layers the strengthening is effective: for instance the strength in uniaxial tension in the direction of the layers is the arithmetic mean of the constituents strengths $v_0 k_0 + v_1 k_1$.

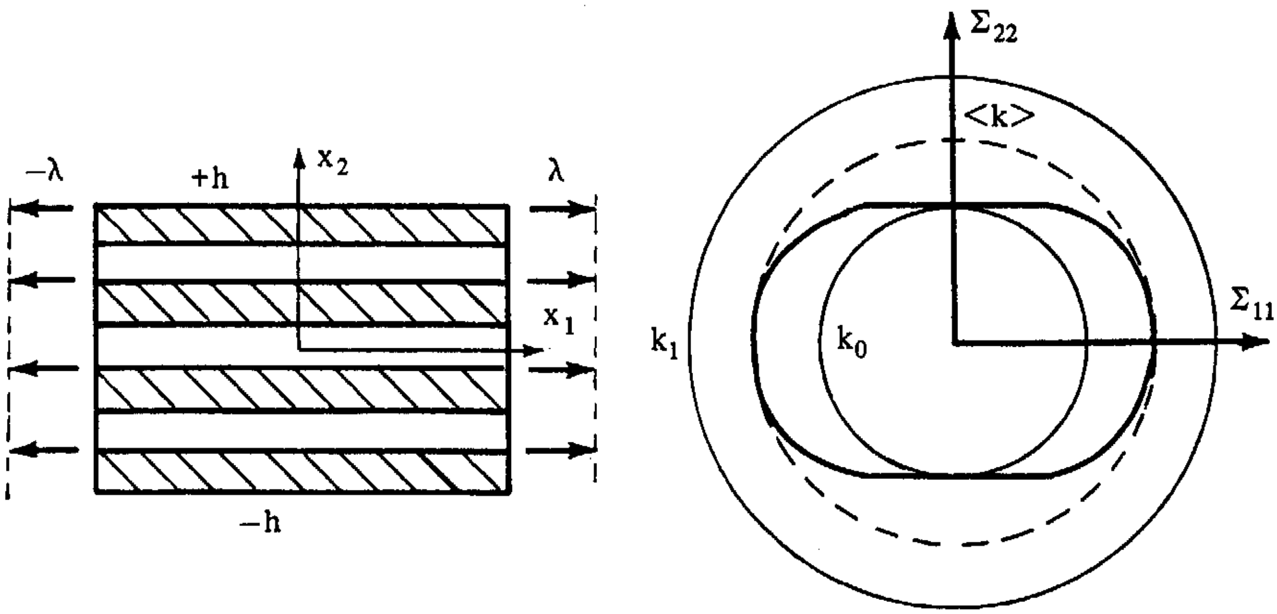

Figure A.1

A.2 An elementary illustration of why the A.V.P. fails

Now consider a rectangular block of this layered material, submitted to uniform tractions parallel to the layers direction:

$$\text{(A.3)} \qquad \begin{cases} \text{for } x_1 = \pm L \ , \ \sigma_{11} = \pm \lambda \ , \ \sigma_{12} = 0, \\ \\ \text{for } x_2 = \pm h \ , \ \sigma_{22} = 0 \ , \ \sigma_{12} = 0 \ . \end{cases}$$

It is readily seen that :

$$\lambda^{\epsilon} = k_0 = \mathrm{Inf}(\, k_0 \,,\, k_1 \,) \quad ,$$

since $|\, \sigma_{11}(x) \,| \leq k_0$ a.e.x in $\Omega_0^{\epsilon} = \left\{ x \in \Omega \,,\, P^{\epsilon}(x) = P_0 \right\}$, $\sigma_{11} = \pm \lambda$ on $x_1 = \pm L$, and since $\partial\Omega_0^{\epsilon}$ intersects $\{ x_1 = \pm L \}$.

It is also readily checked that:

$$\mathrm{Sup}\{\, \lambda \,|\, \Sigma \in P^{hom},\ \mathrm{div}(\Sigma) = 0,\ \Sigma \text{ satisfies (A.3)} \}$$
$$= v_0 k_0 + v_1 k_1 .$$

Therefore $\lim(\lambda^{\epsilon})$ is different from the result provided by the A.V.P..

<u>A.3 Application of Theorem 1</u>

We apply the general result provided by (3.42) for the layered material pictured on Figure A.1 (with strong layers of material 1 at top and bottom of the body), the sets A and B read as:

$$A = \partial\Omega \ ,\ B = \{\, x_1 = \pm L \} \ .$$

Therefore Remark 2 imply that :

$$C(x) = P_0 n(x) \text{ for every } x \text{ in } B.$$

We can state a more specific result, valid for any geometry of the layered material:

$$C(x) = P_0 n(x) \text{ for every } x \text{ in } \Gamma_1 .$$

To prove this affirmation we note that, on A - B, material 1 is exclusively present, and therefore that $n(x) = e_2$, where e_2 gives the direction orthogonal to the layering. According to (A.2) we have:

$$P^{hom} e_2 = \{\, (\Sigma_{12}, \Sigma_{22}) \in \mathbb{R}^2 ,\ \exists\ \sigma(y_2) \in P(y) \text{ a.e. } y \text{ in } Y \,,$$
$$\langle\sigma\rangle = \langle\Sigma\rangle ,\ \sigma_{12} = \Sigma_{12} ,\ \sigma_{22} = \Sigma_{22} \ \} .$$

Note that:

$$P^0 e_2 = \left\{ (\sigma_{12}, \sigma_{22}) \in \mathbb{R}^2 ,\ \exists\ \sigma_{11} \text{ such that } (\sigma_{11},\ \sigma_{12},\ \sigma_{22}) \in P^0 \right\}.$$

Therefore, under the assumption $P^0 \subset P^1$, we have $P^{hom} e_2 = P^0 e_2$. We conclude by means of proposition 2, since:

$$(P^0 \cap P^1) e_2 = P^0 e_2 \ \ , \text{ and } C(x) \cap C^{hom}(x) = P^0 e_2 .$$

We have established the following result: *for a layered material with two constituents such that $P^0 \subset P^1$, the strength on the loaded boundary is ruled by the strength of the weakest constituent.*

We immediately conclude that, in the specific example under consideration, λ^{hom} given by (3.15), or equivalently by the primal characterization (1.10), is equal to k_0 , i.e. to the limit of λ^ϵ. Q.E.D.

Guy BOUCHITTE
U.T.V.
Avenue de l'Université
BP 132. 83957. LA GARDE. Cedex
FRANCE

Pierre SUQUET
L.M.A.
31 Chemin Joseph Aiguier
13402. MARSEILLE. Cedex 09.
FRANCE