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On a $p(t, x)$-Laplace evolution equation with a stochastic force

C. Bauzet · G. Vallet · P. Wittbold · A. Zimmermann

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Abstract In this paper, we are interested in the stochastic forcing of a nonlinear singular/degenerated parabolic problem of $p(t, x)$-Laplace type. Since the Lebesgue and Sobolev spaces with variable exponents of variables t and x are Orlicz type spaces and do not fit into the classical framework of Bochner spaces, we have to adapt to this framework classical methods based on monotonicity arguments.

Keywords $p(t, x)$-Laplace · Variable exponent · Stochastic forcing · Monotonicity

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1 Introduction

In this paper we are interested in the following formal stochastic partial differential problem:

C. Bauzet · G. Vallet (✉)
LMAP - UMR CNRS 5142, IPRA, BP 1155, 64013 Pau Cedex, France
e-mail: guy.vallet@univ-pau.fr

C. Bauzet
e-mail: caroline.bauzet@univ-pau.fr

P. Wittbold · A. Zimmermann
Fakultät für Mathematik, Universität Duisburg-Essen, Universitätsstr. 2, 45141 Essen, Germany
e-mail: petra.wittbold@uni-due.de

A. Zimmermann
e-mail: zimmermann.aleksandra@uni-due.de

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$$(P): \begin{cases} du - \Delta_{p(\cdot)}u\,dt = h(\cdot,u)dw & \text{in } \Omega\times(0,T)\times \mathrm{D} \\ u=0 & \text{on } \Omega\times(0,T)\times\partial\mathrm{D} \\ u(0,\cdot)=u_0 & \text{in } \Omega\times\mathrm{D} \end{cases}$$

with differentiation in the sense of distributions, the equation is equivalent to

$$\partial_t\left(u-\int_0^t h(\cdot,u)dw\right)-\Delta_{p(\cdot)}u=0.$$

In the sequel, we assume that:

- T is a positive number, $\mathrm{D}\subset\mathbb{R}^d$ is a bounded domain with a Lipschitz boundary, $Q_T=(0,T)\times\mathrm{D}$ and $w=\{w_t,\mathcal{F}_t;0\le t\le T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on the classical Wiener space $(\Omega,\mathcal{F},P)$;
- h is a Carathéodory function in the sense that:
  for any $\lambda\in\mathbb{R}$, $h(\cdot,\lambda)\in N^2_W(0,T,L^2(\mathrm{D}))$, the space of predictable processes with values in $L^2(\mathrm{D})$ (see G. Da Prato et al. [4] for example),
  and, $P\otimes\mathcal{L}^{d+1}$-a.e., $\lambda\in\mathbb{R}\to h(t,x,\omega,\lambda)\in\mathbb{R}$ is continuous. Moreover, h is a Lipschitz-continuous function of the variable λ, uniformly with respect to the other variables;
- the variable exponent is a measurable function $p:Q_T\to(1,\infty)$ satisfying $1<p^-=\operatorname{ess\,inf}_{(s,y)\in Q_T}p(s,y)\le p(t,x)\le p^+=\operatorname{ess\,sup}_{(s,y)\in Q_T}p(s,y)<\infty$, and $\Delta_{p(\cdot)}u$ denotes the formal differential operator $\operatorname{div}\left[|\nabla u|^{p(t,x)-2}\nabla u\right]$;
- the initial condition $u_0\in L^2(\mathrm{D})$ and homogeneous Dirichlet boundary conditions are required.

We first explain briefly a physical motivation for such a mathematical study. Concerning the theory of filtration of an elastic fluid in an heterogeneous porous medium (see Barenblatt [2, Sect. 3.2.1]), if one denotes by m the porosity of the medium, ρ the density of the fluid and v the velocity of filtration, the mass conservation equation of the fluid has the form

$$\partial_t(m\rho)+\mathrm{Div}\rho\mathbf{v}=0.$$

The well-known linear Darcy law expresses that the velocity of filtration is proportional to the pressure gradient. In the presence of a non-homogeneous medium or turbulence regime (cf. Diaz et al. [5]), a nonlinear version of the Darcy law may be more appropriate. If moreover the characteristics of the medium may vary in dependence on the spacial zones and evolve during time, the velocity of filtration is then given by the relation $v_i=-k|\nabla P|^{\lambda-2}\nabla P$ where λ is a function of (t,x), k is related to the resistance of the porous medium to the fluid leaking through it and P is the pressure of the fluid.

Following in addition some other physical characteristics in the model [2] and discarding higher-order terms, we find

$$\partial_t P-\kappa\Delta_{p(\cdot)}P=0,\quad \text{where}\quad \kappa=\frac{k}{m_0\beta_f+\beta_r} \tag{1}$$

denotes the coefficient of piezoconductivity.

Then, following Geiger et al. [9], the interpretation of the stochastic integral in the Itô sense $\int_0^t h(\cdots)dw$ comes from the fact that the pore-scale process acts on a time scale that is faster than the macroscopic time scales.

When p is a fixed exponent, Problem (P) is a classical one and we refer to E. Pardoux [14] or to classical books on stochastic partial differential equations, see e.g. Da Prato et al. [4], Grecksch et al. [11] or C. Prévôt et al. [15]. In that case, the solution u belongs a.s. to the Bochner space $L^p(0,T,W_0^{1,p}(\mathrm{D}))$ and $N_W^2(0,T,L^2(\mathrm{D}))$ where $W_0^{1,p}(\mathrm{D})$ denotes the classical Sobolev space and $\partial_t(u-\int_0^t h(\cdot,u)dw)$ to $L^{p'}(0,T,W^{-1,p'}(\mathrm{D}))$ where $p'=\frac{p}{p-1}$ and $W^{-1,p'}(\mathrm{D})$ denotes the dual space of $W_0^{1,p}(\mathrm{D})$.

In these books, the method is based on a monotonicity argument and Minty's trick comes from the possibility to write an Itô formula in this functional setting.

This formula can be proved, on the one hand, by a time-discretization method as in the above cited references or in N. Krylov et al. [13], or, on the other hand, using a regularization with respect to the space variable as in Krylov [12], or Fellah et al. [8].

When p is a variable exponent of the only variable x, let us quote the paper of Ren et al. [16], where the authors consider a problem with values in an Orlicz space. The method is the same, u belongs a.s. to $L^{p^-}(0,T,W_0^{1,p(\cdot)}(\mathrm{D}))$ where $W_0^{1,p(\cdot)}(\mathrm{D})$ will be presented in the next section, and $\partial_t(u-\int_0^t h(\cdot,u)dw)$ to $L^{(p^+)'}(0,T,W^{-1,p'(\cdot)}(\mathrm{D}))$. Here, p^- and $(p^+)'$ are not conjugate exponents, but since the spaces are Bochner spaces, the same method applies.

In the present paper the exponent p is a function of the variables (t,x). Then, if u is a solution, $u(t,\cdot)\in W_0^{1,p(t,\cdot)}(\mathrm{D})$, while $u(s,\cdot)\in W_0^{1,p(s,\cdot)}(\mathrm{D})$ which are not comparable spaces, and the proof of an Itô formula by a time discretization is out of range. Moreover, since the Lebesgue space

$$L^{p(\cdot)}(Q_T):=\left\{u:Q_T\to\mathbb{R},\ \text{measurable},\ \int\limits_{Q_T}|u(t,x)|^{p(t,x)}dxdt<+\infty\right\}$$

is not stable by partial convolution in only the variable x, the second method to prove the Itô formula fails as well; even if each term of the formula (see below) exists and it can be conjectured that u is a.s. continuous with values in $L^2(\mathrm{D})$ and, for any $s<t$,

$$E\|u(t)\|^2_{L^2(\mathrm{D})}-E\|u(s)\|^2_{L^2(\mathrm{D})}+2E\int_s^t\int_{\mathrm{D}}|\nabla u|^{p(t,x)}dxd\sigma=E\int_s^t\int_{\mathrm{D}}|h(u)|^2dxd\sigma.$$

For this reason we had to revisit the method proposed by Pardoux [14].

In the spirit of Bauzet and Vallet [3], where similar questions have been considered for a Barenblatt equation, we first prove the existence of a solution in the additive-noise case by taking advantage of the remark that this type of problem [here (6)] can be reduced to a random one [here (7)]; then we extend the result to a family of multiplicative problems by using Banach fixed-point theorem.

Our steps are the following ones : first, we consider a singular perturbation of Problem (P) with a "nice" function h independent of u and we obtain a stability result of the solution with respect to h; passing to the limit with respect to the singular perturbation, we prove that Problem (P) is well posed for an additive noise if h is a "nice" function, then we prove it for any h in $N_W^2(0, T, L^2(\mathrm{D}))$ by a density argument; in the last step, we solve Problem (P) for a multiplicative noise by a fixed-point argument.

This will be the organization of the paper, just after the following section, devoted to some notations and the presentation of the main result.

2 Preliminaries

For $T > 0$ and $Q_T := (0, T) \times \mathrm{D}$, the variable exponent is a measurable function $p : Q_T \to (1, \infty)$. We will assume that the variable exponent satisfies the following conditions:

- $1 < p^- = \operatorname{ess\,inf}_{(s,y)\in Q_T} p(s, y) \le p(t, x) \le p^+ = \operatorname{ess\,sup}_{(s,y)\in Q_T} p(s, y) < \infty$,
- p is globally log-Hölder continuous, i.e. there exists a constant $c_{log} > 0$ such that
$$|p(t, x) - p(s, y)| \le \frac{c_{log}}{\ln\left(e + \frac{1}{|(t,x)-(s,y)|}\right)}$$
is satisfied for all $(t, x), (s, y) \in Q_T$.

Definition 1 ([6]) For variable exponents we define the variable exponent Lebesgue spaces:

$$L^{p(\cdot)}(Q_T) := \left\{ f : Q_T \to \mathbb{R} \text{ measurable} \mid \int_{Q_T} |f(t, x)|^{p(t,x)} dx dt < \infty \right\}$$

$$L^{p(t,\cdot)}(\mathrm{D}) := \left\{ f : \mathrm{D} \to \mathbb{R} \text{ measurable} \mid \int_{\mathrm{D}} |f(x)|^{p(t,x)} dx < \infty \right\}.$$

$L^{p(\cdot)}(Q_T)$ endowed with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(Q_T)} := \inf \left\{ \lambda > 0 \mid \int_{Q_T} |\lambda^{-1} f(t, x)|^{p(t,x)} dx dt \le 1 \right\}$$

is a uniformly convex and separable Banach space.

Following the ideas of [7], for $t \in (0, T)$ we introduce the following function spaces:

$$V_t(\mathrm{D}) := \left\{ u \in L^2(\mathrm{D}) \cap W_0^{1,1}(\mathrm{D}) \mid \nabla u \in L^{p(t,\cdot)}(\mathrm{D}, \mathbb{R}^d) \right\}.$$

According to [7, Lemma 4.2], $V_t(\mathrm{D})$ endowed with the norm

$$\|u\|_{V_t(\mathrm{D})} := \|u\|_{L^2(\mathrm{D})} + \|\nabla u\|_{L^{p(t,\cdot)}(\mathrm{D},\mathbb{R}^d)}$$

is a separable and reflexive Banach space.

$$X(Q_T) := \left\{u \in L^2(Q_T) \mid \nabla u \in L^{p(\cdot)}(Q_T, \mathbb{R}^d),\ u(t,\cdot) \in V_t(\mathrm{D}) \text{ a.e. } t \in (0,T)\right\}$$

endowed with the norm

$$\|u\|_{X(Q_T)} := \|u\|_{L^2(Q_T)} + \|\nabla u\|_{L^{p(\cdot)}(Q_T,\mathbb{R}^d)}$$

is a separable, reflexive Banach space which is continuously embedded into the Bochner space $L^s(0,T,L^2(\mathrm{D}) \cap W_0^{1,p^-}(\mathrm{D}))$, where $s := \min\{2, p^-\}$. Thanks to the log-Hölder continuity condition on the bounded exponent we get that $\mathcal{D}(Q_T)$ is dense in $X(Q_T)$ (see [7, Theorem 4.7]). The dual space of $X(Q_T)$ can be identified with

$$X'(Q_T) := \left\{T \in \mathcal{D}'(Q_T) \mid T = g - \operatorname{div} G,\ g \in L^2(Q_T),\ G \in L^{p'(\cdot)}(Q_T, \mathbb{R}^d)\right\}.$$

We have the continuous embedding

$$X'(Q_T) \hookrightarrow L^r(0,T,L^2(\mathrm{D}) + W^{-1,(p')^-}(\mathrm{D}))$$

where $r := \min\{2, (p')^-\}$ (see [7, Theorem 5.6. and Remark 5.7]). Since the dual space of $V_t(\mathrm{D})$ can be identified with the space

$$V_t'(\mathrm{D}) := \left\{T \in \mathcal{D}'(D) \mid T = \tilde{g} - \operatorname{div} \tilde{G},\ \tilde{g} \in L^2(\mathrm{D}),\ \tilde{G} \in L^{p'(t,\cdot)}(\mathrm{D}, \mathbb{R}^d)\right\},$$

the duality pairing $\langle \cdot, \cdot \rangle_{X(Q_T)}$ can be written as

$$\begin{aligned}\langle T, \varphi\rangle_{X(Q_T)} &= \int_{Q_T} g\varphi\, dx dt + \int_{Q_T} G \cdot \nabla\varphi\, dx dt \\ &= \int_0^T \langle g(t), \varphi(t)\rangle_{L^2(\mathrm{D})} dt + \int_0^T \langle G(t), \nabla\varphi(t)\rangle_{L^{p(t,\cdot)}(\mathrm{D})} dt \\ &= \int_0^T \langle g(t), \varphi(t)\rangle_{V_t(\mathrm{D})} dt\end{aligned}$$

for any $T = g - \operatorname{div} G \in X'(Q_T)$ and any $\varphi \in X(Q_T)$. Finally, let us introduce the space

$$W(Q_T) := \left\{u \in X(Q_T) \mid u_t \in X'(Q_T)\right\},$$

where u_t is the distributional derivative of u. Endowed with the norm

$$\|u\|_{W(Q_T)} := \|u\|_{X(Q_T)} + \|u_t\|_{X'(Q_T)}$$

it is a Banach space and the set $C^\infty([0, T], \mathcal{D}(D))$ is dense in $W(Q_T)$ (see [7, Theorem 6.6]). Moreover, we have the continuous embedding

$$W(Q_T) \hookrightarrow C([0, T], L^2(\mathrm{D}))$$

and for all $u, v \in W(Q_T)$ and $s, t \in [0, T]$ the following rule of integration by parts holds (see [7, Theorem 7.1]):

$$\langle u_t, \chi_{[s,t]} v\rangle_{X(Q_T)} = \langle u(t), v(t)\rangle_{L^2(\mathrm{D})} - \langle u(s), v(s)\rangle_{L^2(\mathrm{D})} - \langle v_t, \chi_{[s,t]} u\rangle_{X(Q_T)}. \quad (2)$$

Concerning the stochastic framework, we denote by $N_W^2(0, T, L^2(\mathrm{D}))$ the space of the predictable $L^2(\mathrm{D})$-valued processes (cf. [4, p. 94] for example). This space is the space $L^2(]0, T[\times\Omega, L^2(\mathrm{D}))$ for the product measure $dt \otimes dP$ on $\mathcal{P}_T$, the predictable σ-field, for the norm of $L^2(Q_T \times \Omega)$.

Remark 1 Note that for the proofs of all the previous results the log-Hölder continuity of the variable exponent is crucial.

In this paper, we are interested in solutions to the problem (P):

$$\begin{cases} du - \Delta_{p(\cdot)} u\, dt = h(\cdot, u) dw & \text{in } \Omega \times (0, T) \times \mathrm{D} \\ u = 0 & \text{on } \Omega \times (0, T) \times \partial\mathrm{D} \\ u(0, \cdot) = u_0 & \text{in } \Omega \times \mathrm{D}. \end{cases}$$

Definition 2 A solution to Problem (P) is any $u \in L^2[\Omega, C([0, T], L^2(\mathrm{D}))]$ [1] $\cap N_W^2(0, T, L^2(\mathrm{D}))$, a.s. $u(\omega, \cdot) \in X(Q_T)$, such that $u(0, \cdot) = u_0$ with $\nabla u \in L^{p(\cdot)}(\Omega \times Q_T)$[2] satisfying for all $s, t \in [0, T]$, almost surely in Ω, a.e. in D,

$$u(t) - u(s) - \int_s^t \Delta_{p(\sigma,\cdot)} u d\sigma = \int_s^t h(\cdot, u) dw(\sigma)$$

or, equivalently, a.s. in $X'(Q_T)$

$$\partial_t \left[u - \int_0^t h(\cdot, u) dw \right] - \Delta_{p(\cdot)} u = 0.$$

Then, the main result of the paper is

[1] in particular $u \in C([0, T], L^2(\Omega \times \mathrm{D}))$.

[2] in particular $u \in L^\tau(0, T, L^2(\mathrm{D}) \cap W_0^{1,p^-}(\mathrm{D}))$ where $\tau := \min\{2, p^-\}$.

Theorem 1 *There exists a unique solution to Problem (P) in the sense of Definition 2. Moreover, there exists a constant C such that if h_1 and h_2 are two different forces,*

$$
\begin{aligned}
&E\|u_1-u_2\|^2_{C([0,T],L^2(\mathrm{D}))}+E\int_0^T\int_{\mathrm{D}}[|\nabla u_1|^{p(t,x)-2}\nabla u_1\\
&-|\nabla u_2|^{p(t,x)-2}\nabla u_2]\nabla[u_1-u_2]dxdt\\
&\leq C\|h_1(\cdot,u_1)-h_2(\cdot,u_2)\|^2_{L^2(\Omega\times Q_T)}.
\end{aligned}
$$

3 Proof of the main result

The proof of the main result is based on the remark that u is a solution of

$$
\partial_t\left[u-\int_0^t h(\cdot,u)dw\right]-\Delta_{p(\cdot)}u=0
$$

if and only if u is a fixed-point of the application

$$
\mathcal{T}:N^2_W(0,T,L^2(\mathrm{D}))\to N^2_W(0,T,L^2(\mathrm{D})),\ S\mapsto u_S
$$

where u_S is the solution to

$$
\partial_t\left[u-\int_0^t h(\cdot,S)dw\right]-\Delta_{p(\cdot)}u=0.
$$

Assume for the moment that Theorem 1 holds if h is independent of u. Then, the application $\mathcal{T}$ is well-defined. Moreover, if S_1 and S_2 are given in $N^2_W(0,T,L^2(\mathrm{D}))$ and u_{S_1},u_{S_2} are the solutions of $((P),h(\cdot,S_1)),\ ((P),h(\cdot,S_2))$ respectively, then for all $t\in(0,T)$

$$
\begin{aligned}
E\|(u_{S_1}-u_{S_2})(t)\|^2_{L^2(\mathrm{D})}&\leq CE\int_0^t\|h(\cdot,S_1)-h(\cdot,S_2)\|^2_{L^2(\mathrm{D})}ds\\
&\leq CL\int_0^t E\|S_1-S_2\|^2_{L^2(\mathrm{D})}ds, \qquad (3)
\end{aligned}
$$

where L is the Lipschitz constant of h. We fix $\alpha>0$. Multiplying (3) by $e^{-\alpha t}$ and integrating over $(0,T)$ we find

$$
\begin{aligned}
&\int_0^T E\|(u_{S_1}-u_{S_2})(t)\|^2_{L^2(\mathrm{D})}e^{-\alpha t}\,dt\\
&\quad\leq CL\int_0^T\frac{d}{dt}\left(-\frac{1}{\alpha}e^{-\alpha t}\right)\int_0^t E\|S_1-S_2\|^2_{L^2(\mathrm{D})}\,ds\,dt \qquad (4)
\end{aligned}
$$

Using integration by parts on the right-hand side of (4) we obtain

$$\int_0^T E\|(u_{S_1}-u_{S_2})(t)\|^2_{L^2(\mathrm{D})}e^{-\alpha t}\,dt \le \frac{CL}{\alpha}\left(1-e^{-\alpha T}\right)\int_0^T E\|S_1-S_2\|^2_{L^2(\mathrm{D})}e^{-\alpha t}\,dt \tag{5}$$

Choosing $\alpha > 0$ such that $\frac{CL}{\alpha} < 1$ the Banach fixed point theorem and the equivalence of the weighted norm with the L^2-Norm yields the proof of the theorem.

In the next subsections we will prove the theorem when h is not a function of u.

3.1 The additive case for a “nice” h

Consider in this section a function $h \in N^2_W(0,T,H^1_0(\mathrm{D}))$ such that, for any $i = 1,\dots,d$, $\partial_{x_i}h \in L^\infty(\Omega\times Q_T)$. For example assume that $h \in S^2_W(0,T,H^k_0(\mathrm{D}))$, the set of simple predictable processes with values in $H^k_0(\mathrm{D})$ for a sufficiently large value of k of the form $h = \sum_{i=1}^M h_i \sum_{j=1}^{N_i} 1_{\Omega^i_j} 1_{]t^i_j,t^i_{j+1}]}$ where $h_i \in H^k_0(\mathrm{D})$ and $\sum_{j=1}^{N_i} 1_{\Omega^i_j} 1_{]t^i_j,t^i_{j+1}]}$ is a real-valued elementary process (see for example [4,15]).

Thus, since ∂_{x_i} is a continuous linear operator from $H^1_0(\mathrm{D})$ to $L^2(\mathrm{D})$, $\nabla h \in N^2_W(0,T,L^2(\mathrm{D})^d)$ and $\nabla\int_0^t h\,dw = \int_0^t \nabla h\,dw$ a.s.

Moreover, using the Burkholder–Davis–Gundy inequality, for any $r \ge 1$, we arrive at

$$E\left(\left|\nabla\int_0^t h(s,x,\omega)dw(s)\right|^r\right) \le CE\left[\left(\int_0^T |\nabla h(s,x,\omega)|^2\,ds\right)^{r/2}\right] \le C\|\nabla h\|^r_{L^\infty(\Omega\times Q_T)}$$

for any t, x a.e., and thus $\int_0^\cdot h\,dw \in L^r(\Omega\times(0,T),W^{1,r}_0(\mathrm{D}))$ for any r.

For such function h, thanks to Pardoux [14], for any positive ϵ and any real number $q \ge \max(2,p^+)$, there exists a unique solution u^ϵ to the problem

$$du - \epsilon\Delta_q u\,dt - \Delta_{p(\cdot)}u\,dt = h\,dw \tag{6}$$

in $L^q(\Omega, L^q(0,T,W^{1,q}_0(\mathrm{D}))\cap N^2_W(0,T,L^2(\mathrm{D}))$ with $u^\epsilon(t=0)=u_0$.

For convenience, let us systematically denote $v(t) = u(t) - \int_0^t h\,dw$ and remark that u^ϵ is a solution of the above problem, if and only if v^ϵ is a solution of the problem

$$\partial_t v - \epsilon\Delta_q\left[v + \int_0^t h\,dw\right]dt - \Delta_{p(\cdot)}\left[v+\int_0^t h\,dw\right]dt = 0 \tag{7}$$

in the same spaces with the same initial condition.

One deduces first that $\partial_t v^\epsilon \in L^{q'}(\Omega, L^{q'}(0,T,W^{-1,q'}(\mathrm{D}))$; then by the special choice of h in $S^2_W(0,T,H^k_0(\mathrm{D}))$, one has $\int_0^t h\,dw \in L^q(\Omega, L^q(0,T,W^{1,q}_0(\mathrm{D}))$, therefore also $v^\epsilon \in L^q(\Omega, L^q(0,T,W_0^{1,q}(\mathrm{D}))$ and the pivot-space $L^2(\mathrm{D})$ yields $v^\epsilon \in L^2(\Omega, C([0,T], L^2(\mathrm{D})))$ (and u^ϵ as well).

Let us derive now some a priori estimates.

By using Itô's formula to the norm of u^ϵ, one gets, for any t and P a.s.,

$$\|u^\epsilon(t)\|^2_{L^2(\mathrm{D})} - \|u_0\|^2_{L^2(\mathrm{D})} + 2\epsilon\int_0^t\int_{\mathrm{D}} |\nabla u^\epsilon|^q dxds + 2\int_0^t\int_{\mathrm{D}} |\nabla u^\epsilon|^{p(s,x)} dxds$$
$$= 2\int_0^t\int_{\mathrm{D}} hu^\epsilon dxdw + \int_0^t\int_{\mathrm{D}} h^2 dxds.$$

Using Burkholder–Davis–Gundy inequality, one has

$$E\left[\sup_t\left|\int_0^t\int_{\mathrm{D}} hu^\epsilon dxdw\right|\right] \le CE\left[\sqrt{\int_0^T\left(\int_{\mathrm{D}} hu^\epsilon dx\right)^2 ds}\right]$$
$$\le CE\left[\sqrt{\int_0^T\int_{\mathrm{D}} h^2dx\int_{\mathrm{D}}(u^\epsilon)^2 dxds}\right]$$
$$\le CE\left[\sqrt{\sup_t\|u^\epsilon(t)\|^2_{L^2(\mathrm{D})}\int_0^T\|h\|^2_{L^2(\mathrm{D})}ds}\right] = CE\left[\sup_t\|u^\epsilon(t)\|_{L^2(\mathrm{D})}\|h\|_{L^2(Q_T)}\right]$$
$$\le \frac{1}{4}E\left[\sup_t\|u^\epsilon(t)\|^2_{L^2(\mathrm{D})}\right] + C\|h\|^2_{L^2(\Omega\times Q_T)},$$

which yields

$$E\left[\sup_t\|u^\epsilon(t)\|^2_{L^2(\mathrm{D})}\right] + 4\epsilon E\left[\int_{Q_T}|\nabla u^\epsilon|^q dxds\right] + 4E\left[\int_{Q_T}|\nabla u^\epsilon|^{p(s,x)}dxds\right]$$
$$\le C\|h\|^2_{L^2(\Omega\times Q_T)} + 2\|u_0\|^2_{L^2(\mathrm{D})}.$$

Therefore, (u^ϵ) is bounded in $L^\infty(0,T,L^2(\Omega\times\mathrm{D}))$, ∇u^ϵ is bounded in $L^{p(\cdot)}(\Omega\times Q_T)$[3] and $\epsilon E[\int_{Q_T}|\nabla u^\epsilon|^q dxds]$ is bounded as well.

[3] We recall that $\|u\|_{L^{p(\cdot)}(\Omega\times Q_T)} = \inf\{\lambda > 0,\ E[\int_{Q_T}\lambda^{-1}|u(\omega,t,x)|^{p(t,x)}dxdt] \le 1\}$ and that a sequence u_n is bounded in $L^{p(\cdot)}(\Omega\times Q_T)$ if and only if the sequence of modular $\rho_p(u_n) = E[\int_{Q_T}|u_n|^{p(t,x)}dxdt]$ is bounded.

Back on v^ϵ, there exists a full-measure set $\tilde{\Omega}$ in Ω, determined by h, such that for any $\omega \in \tilde{\Omega}$,

$$\frac{1}{2}\frac{d}{dt}\|v^\epsilon\|^2_{L^2(\mathrm{D})} + \epsilon \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{q-2} \nabla\left(v^\epsilon + \int_0^t h dw\right)\cdot \nabla v^\epsilon dx \\ + \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{p(t,x)-2} \nabla\left(v^\epsilon + \int_0^t h dw\right)\cdot \nabla v^\epsilon dx = 0.$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\|v^\epsilon\|^2_{L^2(\mathrm{D})} + \epsilon \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{q} dx + \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{p(t,x)} dx \\ = \epsilon \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{q-2} \nabla\left(v^\epsilon + \int_0^t h dw\right)\cdot \nabla \int_0^t h dw dx \\ + \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{p(t,x)-2} \nabla\left(v^\epsilon + \int_0^t h dw\right)\cdot \nabla \int_0^t h dw dx \\ \leq \epsilon \int_{\mathrm{D}} \frac{1}{q'} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{q} dx + \epsilon \int_{\mathrm{D}} \frac{1}{q} \left|\nabla \int_0^t h dw\right|^{q} dx \\ + \int_{\mathrm{D}} \frac{1}{p'(t,x)} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{p(t,x)} dx + \int_{\mathrm{D}} \frac{1}{p(t,x)} \left|\nabla \int_0^t h dw\right|^{p(t,x)} dx.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|v^\epsilon\|^2_{L^2(\mathrm{D})} + \epsilon \int_{\mathrm{D}} \frac{1}{q} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{q} dx \\ + \int_{\mathrm{D}} \frac{1}{p(t,x)} \left|\nabla\left(v^\epsilon + \int_0^t h dw\right)\right|^{p(t,x)} dx \\ \leq \epsilon \int_{\mathrm{D}} \frac{1}{q} \left|\nabla \int_0^t h dw\right|^{q} dx + \int_{\mathrm{D}} \frac{1}{p(t,x)} \left|\nabla \int_0^t h dw\right|^{p(t,x)} dx, \tag{8}$$

so that v^ϵ is bounded in $C([0,T],L^2(\mathrm{D}))$ and in $X(Q_T)$, and $\epsilon\int_{Q_T}|\nabla(v^\epsilon+\int_0^t hdw)|^q dxdt<Cte$.

By extracting a subsequence, still denoted v^ϵ, there exists $v\in X(Q_T)\cap L^\infty(0,T,L^2(\mathrm{D}))$, weak limit of v^ϵ in $X(Q_T)$ and weak-* in $L^\infty(0,T,L^2(\mathrm{D}))$.

Denote by $\mathbf{A}$ any limit point, for the weak convergence in $L^{p'(\cdot)}(Q_T)$ of $|\nabla(v^\epsilon+\int_0^t hdw)|^{p(\cdot)-2}\nabla(v^\epsilon+\int_0^t hdw)$ and χ a weak limit in $L^2(\mathrm{D})$ of $v^\epsilon(T)$ (up to a subsequence, denoted in the same way if it is needed).

Therefore, integrating (8) and passing to the limit, by weak lower semicontinuity of the modular,

$$\frac{1}{2}\|\chi\|^2_{L^2(\mathrm{D})}+\int_{Q_T}\frac{1}{p(t,x)}\left|\nabla(v+\int_0^t hdw)\right|^{p(t,x)}dxdt\le\frac{1}{2}\|u_0\|^2_{L^2(\mathrm{D})}$$

$$+\int_{Q_T}\frac{1}{p(t,x)}\left|\nabla\int_0^t hdw\right|^{p(t,x)}dxdt.$$

By definition of v^ϵ, for any $\varphi\in\mathcal{D}([0,T]\times\mathrm{D})$,

$$\int_{\mathrm{D}}v^\epsilon(T,x)\varphi(T,x)dx-\int_{Q_T}v^\epsilon(t,x)\partial_t\varphi(t,x)dxdt$$

$$+\epsilon\int_{Q_T}\left|\nabla(v^\epsilon+\int_0^t hdw)\right|^{q-2}\nabla\left(v^\epsilon+\int_0^t hdw\right)\cdot\nabla\varphi dxdt$$

$$+\int_{Q_T}\left|\nabla(v^\epsilon+\int_0^t hdw)\right|^{p(t,x)-2}\nabla\left(v^\epsilon+\int_0^t hdw\right)\cdot\nabla\varphi dxdt=\int_{\mathrm{D}}u_0(x)\varphi(0,x)dx.$$

Then, passing to the limit yields

$$\int_{\mathrm{D}}\chi\varphi(T,x)dx-\int_{Q_T}v(t,x)\partial_t\varphi(t,x)dxdt+\int_{Q_T}\mathbf{A}\cdot\nabla\varphi dxdt=\int_{\mathrm{D}}u_0(x)\varphi(0,x)dx$$

and $\partial_t v-\mathrm{div}\mathbf{A}=0$ in the sense of distributions and $\partial_t v\in X'(Q_T)$.

Since $C^\infty([0,T],\mathcal{D}(\mathrm{D}))$ is dense in $W(Q_T)$ (see [7, Theorem 6.6]), it follows that for all $\varphi\in W(Q_T)$,

$$\int_{\mathrm{D}}\chi(x)\varphi(T,x)dx-\langle\partial_t\varphi,v\rangle+\int_{Q_T}\mathbf{A}\cdot\nabla\varphi dxdt=\int_{\mathrm{D}}u_0(x)\varphi(0,x)dx.$$

Since $\partial_t v \in X'(Q_T)$ and $v \in X(Q_T)$, v is an element of $W(Q_T)$ and, the formula of integration by parts in time ([7]) leads for any $\varphi \in W(Q_T)$,

$$\int_{\mathrm{D}} [\chi(x) - v(T,x)]\varphi(T,x)dx + \langle \partial_t v, \varphi\rangle + \int_{Q_T} \mathbf{A} \cdot \nabla\varphi dxdt$$
$$= \int_{\mathrm{D}} [u_0(x) - v(0,x)]\varphi(0,x)dx,$$

or

$$\int_{\mathrm{D}} [\chi(x) - v(T,x)]\varphi(T,x)dx + \langle \partial_t v - \mathrm{div}\mathbf{A}, \varphi\rangle = \int_{\mathrm{D}} [u_0(x) - v(0,x)]\varphi(0,x)dx.$$

Therefore

$$\int_{\mathrm{D}} [\chi(x) - v(T,x)]\varphi(T,x)dx = \int_{\mathrm{D}} [u_0(x) - v(0,x)]\varphi(0,x)dx$$

and one gets that $v(T) = \chi$ and $v(0) = u_0$. Note that in our reasoning T is arbitrary, so that it is possible to conclude, for any t, that $v^\epsilon(t)$ converges weakly to $v(t)$ in $L^2(\mathrm{D})$.

Let us also mention the additional information: since $v \in W(Q_T)$ one has that $v \in C([0,T], L^2(\mathrm{D}))$ and, for any fixed t in the sequel, the following energy equality holds

$$\frac{1}{2}\|v(t)\|^2_{L^2(\mathrm{D})} + \int_{Q_t} \mathbf{A} \cdot \nabla v dxds = \frac{1}{2}\|u_0\|^2_{L^2(\mathrm{D})}. \tag{9}$$

Moreover, since

$$\frac{1}{2}\frac{d}{dt}\|v^\epsilon\|^2_{L^2(\mathrm{D})} + \epsilon\int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t hdw\right)\right|^q dx + \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t hdw\right)\right|^{p(t,x)} dx$$
$$= \epsilon\int_{\mathrm{D}} \left|\nabla(v^\epsilon + \int_0^t hdw)\right|^{q-2} \nabla\left(v^\epsilon + \int_0^t hdw\right) \cdot \nabla\int_0^t hdwdx$$
$$+ \int_{\mathrm{D}} \left|\nabla\left(v^\epsilon + \int_0^t hdw\right)\right|^{p(t,x)-2} \nabla\left(v^\epsilon + \int_0^t hdw\right) \cdot \nabla\int_0^t hdwdx,$$

the following estimate holds

$$\frac{1}{2}\|v^\epsilon(t)\|^2_{L^2(\mathrm{D})}+\int_{Q_t}\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{p(s,x)}dxds \tag{10}$$
$$\leq \epsilon\int_{Q_t}\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{q-2}\nabla\left(v^\epsilon+\int_0^s hdw\right)\cdot\nabla\int_0^s hdwdxds$$
$$+\int_{Q_t}\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{p(s,x)-2}\nabla\left(v^\epsilon+\int_0^s hdw\right)\cdot\nabla\int_0^s hdwdxds$$
$$+\frac{1}{2}\|u_0\|^2_{L^2(\mathrm{D})},$$

and, at the limit with respect to ϵ, one gets

$$\frac{1}{2}\|v(t)\|^2_{L^2(\mathrm{D})}+\limsup_\epsilon\int_{Q_t}\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{p(s,x)}dxds$$
$$\leq\int_{Q_t}\mathbf{A}\cdot\nabla\int_0^s hdwdxds+\frac{1}{2}\|u_0\|^2_{L^2(\mathrm{D})}.$$

Using (9), one has

$$\limsup_\epsilon\int_{Q_t}\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{p(s,x)}dxds\leq\int_{Q_t}\mathbf{A}\cdot\nabla\left(v+\int_0^s hdw\right)dxds$$

and, since $|\nabla(v^\epsilon+\int_0^s hdw)|^{p(s,x)-2}\nabla(v^\epsilon+\int_0^s hdw)$ converges weakly to $\mathbf{A}$ in $L^{p'(\cdot)}(Q_T)$ and ∇v^ϵ converges weakly to ∇v in $L^{p(\cdot)}(Q_T)^d$, one concludes that

$$\limsup_\epsilon\int_{Q_t}\left[\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{p(s,x)-2}\nabla\left(v^\epsilon+\int_0^s hdw\right)-\mathbf{A}\right]\cdot\nabla(v^\epsilon-v)dxds$$
$$=\limsup_\epsilon\int_{Q_t}\left[\left|\nabla(v^\epsilon+\int_0^s hdw)\right|^{p(s,x)-2}\nabla\left(v^\epsilon+\int_0^s hdw\right)\right].$$

$$\nabla\left(v^\epsilon+\int_0^s hdw-v-\int_0^s hdw\right)dxds$$

$$=\limsup_\epsilon\int_{Q_t}\left|\nabla\left(v^\epsilon+\int_0^s hdw\right)\right|^{p(s,x)}dxds-\int_{Q_t}\mathbf{A}\cdot\nabla\left(v+\int_0^s hdw\right)dxds\le 0.$$

Let us denote by $J:X(Q_T)\to\mathbb{R}$, $u\mapsto\int_{Q_T}\frac{1}{p(t,x)}|\nabla u(t,x)|^{p(t,x)}dxdt$. J is a proper convex function; moreover, it is continuous and Gâteaux-differentiable. Thus, its Gâteaux-derivative is a single-valued maximal monotone operator from $X(Q_T)$ to its dual space.

By construction, $DJ=-\Delta_{p(\cdot)}$ and, for any (u,v) in $X(Q_T)$, one has:

$$\langle DJ(u),v\rangle_{X(Q_T)}=\int_{Q_T}|\nabla u|^{p(t,x)-2}\nabla u\nabla v dxdt.$$

In terms of DJ the preceding estimate reads as

$$\limsup_\epsilon\left\langle DJ\left(v^\epsilon+\int_0^s hdw\right)+\mathrm{div}\mathbf{A},v^\epsilon-v\right\rangle_{X(Q_T)}\le 0$$

and the properties of maximal monotone operators in reflexive Banach spaces (cf. e.g. Barbu [1]) ensure that $DJ(v+\int_0^t hdw)=-\mathrm{div}\mathbf{A}$ and then that $\langle DJ(v^\epsilon+\int_0^s hdw),v^\epsilon+\int_0^s hdw\rangle_{X(Q_T)}\to\langle DJ(v+\int_0^s hdw),v+\int_0^s hdw\rangle_{X(Q_T)}$. Therefore, the remark saying that in $L^{p(\cdot)}(Q_T)$ weak convergence of a sequence plus the convergence of the modular implies the strong convergence (Giacomoni et al. [10, Appendix]) leads to the convergence, on the one hand of $\nabla(v^\epsilon+\int_0^t hdw)$ to $\nabla(v+\int_0^t hdw)$, then, on the other hand of ∇v^ϵ to ∇v, in $L^{p(\cdot)}(Q_T)$.

Using that $\mathbf{A}=|\nabla(v+\int_0^t hdw)|^{p(t,x)-2}\nabla(v+\int_0^t hdw)$, the limit superior in (10) yields

$$\frac{1}{2}\limsup_\epsilon\|v^\epsilon(t)\|^2_{L^2(\mathrm{D})}+\int_{Q_t}\left|\nabla\left(v+\int_0^s hdw\right)\right|^{p(s,x)}dxds$$

$$\le\int_{Q_t}\left|\nabla\left(v+\int_0^s hdw\right)\right|^{p(s,x)-2}\nabla\left(v+\int_0^s hdw\right)\cdot\nabla\int_0^s hdwdxds+\frac{1}{2}\|u_0\|^2_{L^2(\mathrm{D})}.$$

Then, since replacing $\mathbf{A}$ by its value in (9) gives

$$\frac{1}{2}\|v(t)\|^2_{L^2(\mathrm{D})} + \int_{Q_t} \left|\nabla(v + \int_0^s hdw)\right|^{p(s,x)-2} \nabla\left(v + \int_0^s hdw\right) \cdot \nabla v dxds$$
$$= \frac{1}{2}\|u_0\|^2_{L^2(\mathrm{D})}, \tag{11}$$

one gets that

$$\limsup_{\epsilon} \|v^\epsilon(t)\|^2_{L^2(\mathrm{D})} \leq \|v(t)\|^2_{L^2(\mathrm{D})},$$

and since we had previously a weak convergence in $L^2(\mathrm{D})$ of the sequence, one concludes, for any t, that $v^\epsilon(t)$ converges to $v(t)$ in $L^2(\mathrm{D})$, and then of v^ϵ to v in $L^2(Q_T)$, thus, in particular the convergence of v^ϵ to v in $X(Q_T)$.

Finally, since v is a solution of a problem that admits a unique solution, the above convergence result is available for the whole sequence v^ϵ and not for a subsequence as previously claimed.

Now, we want to show the convergence of v^ϵ to v in $L^2(\Omega \times Q_T)$ and of ∇v^ϵ to ∇v in $L^{p(\cdot)}(\Omega \times Q_T)$.

We have already shown that $v_\epsilon(\omega) \to v(\omega)$ in $L^2(Q_T)$ almost surely in Ω when $\epsilon \downarrow 0$.

From (8) it follows that for all $t \in [0, T]$ a.s. in Ω

$$\|v^\epsilon(t)\|^2_{L^2(\mathrm{D})} + \frac{2\epsilon}{q} \int_{Q_t} \left|\nabla\left(v^\epsilon + \int_0^\sigma hdw\right)\right|^q dxd\sigma$$
$$+\frac{2}{p^+} \int_{Q_t} \left|\nabla\left(v^\epsilon + \int_0^\sigma hdw\right)\right|^{p(\sigma,x)} dxd\sigma$$
$$\leq 2\epsilon \int_{Q_T} \frac{1}{q}\left|\nabla \int_0^\sigma hdw\right|^q dxd\sigma + \frac{2}{p^-}\int_{Q_T} \left|\nabla \int_0^\sigma hdw\right|^{p(\sigma,x)} dxd\sigma + \|u_0\|^2_{L^2(\mathrm{D})}, \tag{12}$$

and that a.s. in Ω we have

$$\int_{Q_T} \left|\nabla \int_0^\sigma hdw\right|^{p(\sigma,x)} dxd\sigma \leq C(|Q_T|) + \int_{Q_T} \left|\nabla \int_0^\sigma hdw\right|^{p^+} dxd\sigma,$$

with $C(|Q_T|) > 0$ not depending on ω, hence v_ϵ converges to v in $L^2(\Omega \times Q_T)$ using the regularity of h and the Lebesgue's dominated convergence theorem.

For the convergence of ∇v^ϵ to ∇v in $L^{p(t,x)}(\Omega \times Q_T)$, we need to show that

$$\int_{\Omega}\int_{Q_T} |\nabla(v^\epsilon - v)|^{p(t,x)} dxdtdP(\omega) \to 0 \text{ for } \epsilon \to 0.$$

We already know that, almost surely in Ω

$$\int_{Q_T} |\nabla(v^\epsilon(\omega) - v(\omega))|^{p(t,x)} dxdt \to 0 \text{ for } \epsilon \to 0,$$

and that, from Young's inequality, (11) and (12), there exists a positive constant $C(p^+, p^-)$, such that

$$\begin{aligned}
&\int_{Q_T} |\nabla(v^\epsilon(\omega) - v(\omega))|^{p(t,x)} dxdt \\
&\leq 2^{p^+}\left[\int_{Q_T}\left|\nabla\left(v^\epsilon(\omega)+\int_0^t hdw\right)\right|^{p(t,x)} dxdt+\int_{Q_T}\left|\nabla\left(v(\omega)+\int_0^t hdw\right)\right|^{p(t,x)} dxdt\right] \\
&\leq C(p^+, p^-)\left(\int_{Q_T}\left|\nabla\int_0^\sigma hdw\right|^q dxd\sigma+\int_{Q_T}\left|\nabla\int_0^\sigma hdw\right|^{p(\sigma,x)} dxd\sigma+\|u_0\|^2_{L^2(\mathrm{D})}\right)
\end{aligned}$$

for all $0 < \epsilon < 1$ and the assertion follows using the regularity of h and Lebesgue's dominated convergence theorem.

Since the same convergences hold for u^ϵ to $u = v - \int_0^t hdw$ and since u^ϵ is a predictable process with values in $L^2(\mathrm{D})$, this is the same for u.

3.2 The additive case for a general h

Applying Itô's formula to u^ϵ, we get, for any t,

$$\begin{aligned}
&\|u^\epsilon(t)\|^2_{L^2(\mathrm{D})} - \|u_0\|^2_{L^2(\mathrm{D})} + 2\epsilon\int_0^t\int_{\mathrm{D}} |\nabla u^\epsilon|^q dxds + 2\int_0^t\int_{\mathrm{D}} |\nabla u^\epsilon|^{p(s,x)} dxds \\
&= 2\int_0^t\int_{\mathrm{D}} hu^\epsilon dxdw + \int_0^t\int_{\mathrm{D}} h^2 dxds,
\end{aligned}$$

and, passing to the limit,

$$\begin{aligned}
&\|u(t)\|^2_{L^2(\mathrm{D})} - \|u_0\|^2_{L^2(\mathrm{D})} + 2\int_{Q_t} |\nabla u|^{p(s,x)} dxds \\
&\leq 2\int_{Q_t} hudxdw + \int_{Q_t} h^2 dxds.
\end{aligned} \tag{13}$$

Moreover, if h_i, $i = 1, 2$, are two elements associated to u_i^ϵ and to u_i, one gets

$$\|(u_1^\epsilon - u_2^\epsilon)(t)\|^2_{L^2(\mathrm{D})} + 2\epsilon \int_0^t \int_{\mathrm{D}} [|\nabla u_1^\epsilon|^{q-2}\nabla u_1^\epsilon - |\nabla u_2^\epsilon|^{q-2}\nabla u_2^\epsilon]\nabla[u_1^\epsilon - u_2^\epsilon]dxds$$

$$+2\int_0^t \int_{\mathrm{D}} [|\nabla u_1^\epsilon|^{p(s,x)-2}\nabla u_1^\epsilon - |\nabla u_2^\epsilon|^{p(s,x)-2}\nabla u_2^\epsilon]\nabla[u_1^\epsilon - u_2^\epsilon]dxds$$

$$= 2\int_0^t \int_{\mathrm{D}} [h_1 - h_2][u_1^\epsilon - u_2^\epsilon]dxdw + \int_0^t \int_{\mathrm{D}} [h_1 - h_2]^2 dxds,$$

and, passing to the limit

$$\|(u_1 - u_2)(t)\|^2_{L^2(\mathrm{D})} + 2\int_0^t \int_{\mathrm{D}} [|\nabla u_1|^{p(s,x)-2}\nabla u_1$$
$$-|\nabla u_2|^{p(s,x)-2}\nabla u_2]\nabla[u_1 - u_2]dxds$$
$$\leq 2\int_0^t \int_{\mathrm{D}} [h_1 - h_2][u_1 - u_2]dxdw + \int_0^t \int_{\mathrm{D}} [h_1 - h_2]^2 dxds, \tag{14}$$

A consequence is the Lipschitz continuity of the application

$$T : S^2_W(0, T, H^k_0(\mathrm{D})) \to N^2_W(0, T, L^2(\mathrm{D})),\ h \mapsto u.$$

Then, the density of $S^2_W(0, T, H^k_0(\mathrm{D}))$ in the Hilbert space $N^2_W(0, T, L^2(\mathrm{D}))$ ensures the existence and uniqueness of the extension $\bar{T}$ of T from $N^2_W(0, T, L^2(\mathrm{D}))$ into itself. This gives a result of existence and uniqueness of an element called a mild solution when h belongs to $N^2_W(0, T, L^2(\mathrm{D}))$.

Denote by u a mild solution associated with h, $(h_m) \subset S^2_W(0, T, H^k_0(\mathrm{D}))$ a sequence converging to h in $N^2_W(0, T, L^2(\mathrm{D}))$ and (u_m) the sequence of the corresponding solutions.

Using (14) and Burkholder–Davis–Gundy inequality, we deduce

$$E\int_0^T \int_{\mathrm{D}} [|\nabla u_n|^{p(t,x)-2}\nabla u_n - |\nabla u_m|^{p(t,x)-2}\nabla u_m]\nabla[u_n - u_m]dxdt$$
$$+E\|u_n - u_m\|^2_{C([0,T],L^2(\mathrm{D}))} \leq C\|h_n - h_m\|^2_{L^2(\Omega\times Q_T)},$$

which implies that

$$E\int_0^T \int_{\mathrm{D}} [|\nabla u_n|^{p(t,x)-2}\nabla u_n - |\nabla u_m|^{p(t,x)-2}\nabla u_m]\nabla[u_n - u_m]dxdt$$

tends to 0 when m and n tend to infinity, and that u_m is a Cauchy sequence in $L^2(\Omega, C([0,T], L^2(\mathrm{D})))$.

This allows us to conclude that the mild solution u is a continuous process with values in $L^2(\mathrm{D})$.

It remains to prove that u is a solution in the sense of Definition 2. To this end denote by $\mathcal{J}$ the proper convex, continuous and Gâteaux differentiable mapping, defined on $\mathcal{E} := \{u \in N^2_W(0,T,L^2(\mathrm{D})),\ \nabla u \in L^{p(\cdot)}(\Omega \times Q_T)\}$ endowed with the norm $\|u\| = \|u\|_{N^2_W(0,T,L^2(\mathrm{D}))} + \|\nabla u\|_{L^{p(\cdot)}(\Omega\times Q_T)}$, by $\mathcal{J}(u) = E\int_{Q_T}[\frac{u^2}{2} + \frac{1}{p(t,x)}|\nabla u|^{p(t,x)}]dxdt$.

As previously mentioned in a similar case its Gâteaux-derivative $D\mathcal{J}(u)$ satisfies, for any $v \in \mathcal{E}$, $\langle D\mathcal{J}(u), v\rangle = E\int_{Q_T} uv + |\nabla u|^{p(t,x)-2}\nabla u\cdot\nabla v dxdt$, is a single-valued maximal monotone operator and the above estimate ensures that

$$\lim_{n,m\to\infty}\langle D\mathcal{J}(u_n) - D\mathcal{J}(u_m), u_n - u_m\rangle = 0.$$

From (13), the sequence (u_n) is bounded in the reflexive Banach space $\mathcal{E}$, so that a subsequence (u_{n_l}) converges weakly in $\mathcal{E}$ to the only possible limit u, and $D\mathcal{J}(u_n)$ converges weakly in the dual space.

The classical results of the theory of maximal monotone operators in reflexive spaces implies that the weak limit has to be $D\mathcal{J}(u)$ and that $\langle D\mathcal{J}(u_n), u_n\rangle$ tends to $\langle D\mathcal{J}(u), u\rangle$.

Again, weak convergence plus convergence of the modular in $L^{p(\cdot)}(\Omega\times Q_T)$ yields the strong convergence of u_{n_l} to u in $\mathcal{E}$.

Finally, the uniqueness of the possible limit ensures that the convergence holds for the whole sequence (u_n). It is therefore possible to pass to the limit in all the terms of the equation satisfied by u_n and conclude that the mild solution u is also a solution in the sense of Definition 2.

Moreover, if u_1 and u_2 are two given solutions, for the same initial condition u_0, then, a.s.

$$\partial_t[u_1 - u_2] - [\Delta_{p(\cdot)}u_1 - \Delta_{p(\cdot)}u_2] = 0,$$

and, by testing this problem with the test-function $u_1 - u_2$, one gets that

$$\|u_1(t) - u_2(t)\|^2_{L^2(\mathrm{D})} + \int_{Q_t}\left[|\nabla u_1|^{p(s,x)-2}\nabla u_1 - |\nabla u_2|^{p(s,x)-2}\nabla u_2\right]\cdot \nabla(u_1 - u_2)dxds = 0,$$

and the weak solution is unique.

Now, coming back to the introduction of Sect. 3, one is able to conclude that Theorem 1 is proved.

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References

1. Barbu, V.: Nonlinear Differential Equations of Monotone Types in Banach Spaces. Springer Monographs in Mathematics. Springer, New York (2010)
2. Barenblatt, G.I.: Similarity, Self-Similarity, and Intermediate Asymptotics. Consultants Bureau, New York (1979)
3. Bauzet, C., Vallet, G.: On abstract Barenblatt equations. Differ. Equ. Appl. **3**(4), 487–502 (2011)
4. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, vol. 44 of Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (1992)
5. Diaz, J.I., De Thélin, F.: On a nonlinear parabolic problem arising in some models related to turbulent flows. SIAM J. Math. Anal. **25**, 1085–1111 (1994)
6. Diening, L., Harjulehto, P., Hästö, P., Ruzicka, M.: Lebesgue and Sobolev Spaces with Variable Exponents. Springer, Heidelberg (2011)
7. Diening, L., Nägele, P., Ruzicka, M.: Monotone operator theory for unsteady problems in variable exponent spaces. Complex Variables Elliptic Equ. **57**, 1209–1231 (2012)
8. Fellah, D., Pardoux, E.: Une formule d'Itô dans des espaces de Banach, et application. In: Stochastic Analysis and Related Topics (Silivri, 1990), vol. 31 of Progr. Probab., pp. 197–209. Birkhäuser, Boston (1992)
9. Geiger, S., Lord, G., Tambue, A.: Exponential time integrators for stochastic partial differential equations in 3D reservoir simulation. Comput. Geosci. **16**, 323–334 (2012)
10. Giacomoni, J., Vallet, G.: Some results about an anisotropic $p(x)$-Laplace-Barenblatt equation. Adv. Nonlinear Anal. **1**(3), 277–298 (2012)
11. Grecksch, W., Tudor, C.: Stochastic Evolution Equations : A Hilbert Space Approach, vol. 85 of Mathematical Research. Akademie-Verlag, Berlin (1995)
12. Krylov, N.V.: Itô's formula for the L_p-norm of stochastic W_p^1-valued processes. Probab. Theory Relat. Fields **147**(3–4), 583–605 (2010)
13. Krylov, N.V., Rozovskii, B.L.: Stochastic evolution equations. In: Stochastic Differential Equations: Theory and Applications, vol. 2 of Interdiscip. Math. Sci., pp. 1–69. World Scientific Publications, Hackensack (2007)
14. Pardoux, E.: Équations aux dérivées partielles stochastiques non linéaires monotones. PhD Thesis, Paris Sud-Orsay (1975)
15. Prévôt, C., Röckner, M.: A concise Course on Stochastic Partial Differential Equations, vol. 1905 of Lecture Notes in Mathematics. Springer, Berlin (2007)
16. Ren, J., Röckner, M., Wang, F.-Y.: Stochastic generalized porous media and fast diffusion equations. J. Differ. Equ. **238**(1), 118–152 (2007)