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THE CAUCHY PROBLEM FOR CONSERVATION LAWS WITH A MULTIPLICATIVE STOCHASTIC PERTURBATION

CAROLINE BAUZET* and GUY VALLET†

UMR - CNRS 5142, IPRA BP 1155
64013 Pau Cedex France
*caroline.bauzet@etud.univ-pau.fr
†guy.vallet@univ-pau.fr

PETRA WITTBOLD

Fakultät für Mathematik, Universitätsstr. 2
45141 Essen, Germany
petra.wittbold@uni-due.de

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Abstract. We study the Cauchy problem for multi-dimensional nonlinear conservation laws with multiplicative stochastic perturbation. Using the concept of measure-valued solutions and Kruzhkov's entropy formulation, the existence and uniqueness of an entropy solution is established.

Keywords: Stochastic PDE; first-order hyperbolic equation; Cauchy problem; multiplicative stochastic perturbation; Young measures; Kruzhkov's entropy.

Mathematics Subject Classification 2010: 35R60, 35L60, 60H15

1. Introduction

In this paper, we are interested in the formal stochastic nonlinear conservation law of type:

$$du - \operatorname{div}(\vec{\mathbf{f}}(u))dt = h(u)dw \quad \text{in } \Omega \times \mathbb{R}^d \times]0, T[, \tag{1.1}$$

with an initial condition u_0 and $d \geq 1$.

In the sequel we assume that T is a positive number, $Q =]0, T[\times \mathbb{R}^d$ and that $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on the classical Wiener space $(\Omega, \mathcal{F}, P)$. These assumptions on W are made for convenience.

Let us assume that

H_1: $\vec{\mathbf{f}} = (f_1, \ldots, f_d) : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz-continuous function and $\vec{\mathbf{f}}(0) = 0$.
H_2: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$.
H_3: $u_0 \in L^2(\mathbb{R}^d)$.

Our aim is to prove a result of existence and uniqueness of the stochastic entropy solution to the above-mentioned problem.

Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. One needs to introduce the notion of entropy solution in order to discriminate the physical solution. Our aim is to adapt the known methods for first-order nonlinear PDE to noise perturbed ones.

Many papers on the viscous parabolic Burgers type stochastic problem can be found in the literature. Let us mention, without exhaustiveness, Da Prato, Debussche and Temam [5], Da Prato and Zabczyk [6], Grecksch and Tudor [11] or Gyöngy and Nualart [12].

Only few papers have been devoted to the study of stochastic perturbation of nonlinear first-order hyperbolic problems. Most of them are interested in the Cauchy problem in the 1D case and/or in the case of additive noise (i.e. with a right-hand side $h(t,x)dw$ independent of the solution u). Let us cite the paper of Holden *et al.* [13] where an operator splitting method is proposed to prove the existence of a weak solution to the Cauchy problem

$$du + f(u)_x dt = g(u)dw \quad \text{in }]0,T[\times \mathbb{R}.$$

The convergence is obtained by using path-wise arguments.

In the paper of Weinan, Khanin and Sinai [26], the authors are interested in the invariant measures for the Burgers equation

$$du + \frac{1}{2}(u^2)_x dt = \left(\sum_{k\geq 0} F_k(x)dw_k\right)_x \qquad \text{in }]0,T[\times \mathbb{R}$$

with a periodic assumption in space. Existence and uniqueness of a stochastic entropy solution is proved thanks to a Hopf–Lax type formula for the corresponding stochastic Hamilton–Jacobi equation.

In the paper of Kim [14] a method of compensated compactness is used to prove, via vanishing viscosity approximation, existence of a stochastic weak entropy solution to the Cauchy problem

$$du + f(u)_x dt = h(t,x)dw \quad \text{in }]0,T[\times \mathbb{R}.$$

A Kruzhkov-type method is used to prove the uniqueness.

In [25], Vallet and Wittbold proposed to extend the result of Kim to the multi-dimensional Dirichlet problem for a nonlinear conservation law with additive noise

$$du + \operatorname{div} \vec{\mathbf{f}}(u)dt = h(t,x)dw \ \text{ in }]0,T[\times D; \quad \text{“}u = 0 \ \text{ on }]0,T[\times\partial D\text{”},$$

where D is a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$.

As weak and entropy solutions are not smooth enough allowing for trace properties and, moreover, a Dirichlet condition can only be imposed on the free set of entering characteristics, the boundary condition has to be understood in an appropriate way. In [25] the authors followed the approach of Carrillo which consists in formulating the boundary condition implicitly via global integral entropy inequalities involving the semi-Kruzhkov entropies. Using the vanishing viscosity method and Young measure techniques the authors proved existence, and, via Kruzhkov doubling variables technique, the uniqueness of the stochastic entropy solution.

Feng and Nualart [10] proposed an extension of Kim's result in another direction, namely to the Cauchy problem in $\mathbb{R}^d$

$$du + \operatorname{div} \vec{\mathbf{f}}(u)dt = \int_{z\in Z} \sigma(\cdot, u, z)dw(t, z)$$

with multiplicative noise. The dependence of the right-hand side on u leads to considerable new difficulties. Indeed, in the case of additive noise $h(t,x)dw$ the equation, which has to be understood in the following way

$$\partial_t \left[u - \int_0^t hdw(s) \right] - \operatorname{div} \vec{\mathbf{f}}(u) = 0 \quad \text{in } \mathcal{D}'(Q),$$

can be formulated, via the change of variables $v = u - \int_0^t hdw(s)$, as the random problem:

$$\partial_t v + \operatorname{div} \vec{\mathbf{f}}(\omega, t, x, v) = 0$$

with a flux function $\vec{\mathbf{f}}(\omega, t, x, v) = \vec{\mathbf{f}}(v - \int_0^t hdw(s))$. In this equation, the stochastic variable ω, at least formally, only plays the role of a parameter and thus essentially deterministic techniques can be applied (though it is not possible to use exclusively path-wise arguments).

In presence of multiplicative noise, problem $\partial_t[u - \int_0^t h(u)dw(s)] - \operatorname{div} \vec{\mathbf{f}}(u) = 0$ is nonlocal in time and, mainly due to the lack of regularity of u, a similar reduction is not possible. For this reason, Feng and Nualart introduced in [10] a notion of strong entropy solution in order to prove uniqueness of the entropy solution.

Using the vanishing viscosity and compensated compactness arguments, the authors established existence of strong entropy solutions only in the 1D case.

In the recent paper [4], Chen, Ding and Karlsen propose to revisit the work of Feng and Nualart. They prove that the multi-dimensional stochastic problem is well-posed by using a uniform spatial BV-bound. They prove the existence of strong stochastic entropy solutions in $L^p \cap BV$ and develop a "continuous dependence" theory for stochastic entropy solutions in BV, which can be used to derive an error estimate for the vanishing viscosity method.

Finally, let us mention the paper by Debussche and Vovelle [7] concerning the d-dimensional problem with multiplicative noise

$$du + f(u)_x dt = h(u)dw$$

which is considered on a torus. The authors use the kinetic formulation of the problem and prove existence and uniqueness of a kinetic solution.

In this paper, we propose to prove a result on existence and uniqueness of a stochastic entropy solution, in the sense of Definition 2.2, to the Cauchy problem for the stochastic conservation law with multiplicative noise (1.1) in the d-dimensional case, and with weaker assumptions than the above authors. A method of artificial viscosity is proposed to prove the existence of a solution. The compactness properties used are based on the theory of Young measures and on measure-valued solutions. In particular, for the convergences, instead of path-wise arguments and adapted processes, we propose to use the topological properties of the L^2-type Lebesgue space when it is endowed with the predictable σ-algebra.

Then, an appropriate adaptation of Kruzhkov's doubling variables technique is proposed to prove that any stochastic entropy solution is equal to the one given by the artificial viscosity method. The notion of strong entropy condition of Feng and Nualart does not seem to be required to capture the noise–noise interaction, instead we recover that term through a comparison with the vanishing viscosity solution. Thus, the entropy inequalities are sufficient to prove uniqueness via Kato-type inequalities. This yields the uniqueness of the measure-valued entropy solution, and, by standard arguments, this allows to deduce existence and uniqueness of the stochastic weak entropy solution.

The paper is organized as follows. In Sec. 2, we introduce the notion of stochastic entropy (respectively, measure-valued entropy) solution for (1.1) and establish some basic properties of such solutions. In Sec. 3, existence of a measure-valued entropy solution for (1.1) is proved via a vanishing viscosity approximation. Section 4 is devoted to the proof of uniqueness and of a contraction principle for measure-valued solutions. As a by-product we deduce existence and uniqueness of the entropy solution of the Cauchy problem for (1.1). In the appendix we have collected several auxiliary results (e.g. on regularity of stochastic integrals with respect to parameters, some basic results from Young measure theory and, for the convenience of the reader, we have also included a proof of existence of weak solutions for the approximate viscous parabolic problem).

First of all, we need to introduce some notations and make precise the functional setting.

Notations: In the sequel we denote by $H^1(\mathbb{R}^d)$ the usual Sobolev space.

We recall that $H^1(\mathbb{R}^d)$ is also the closure of $\mathcal{D}(\mathbb{R}^d)$, the space of $C^\infty(\mathbb{R}^d)$-functions with compact support in $\mathbb{R}^d$. We denote by $H^{-1}(\mathbb{R}^d)$ the dual space of $H^1(\mathbb{R}^d)$ which is also the space of derivatives of order less than one of elements of $L^2(\mathbb{R}^d)$ in the common Gelfand–Lions identification $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \equiv L^2(\mathbb{R}^d)' \hookrightarrow H^1(\mathbb{R}^d)'$.

For any positive M, denote by $Q_M =]0,T[\times B(0,M)$ where $B(0,M)$ is the bounded open ball in $\mathbb{R}^d$ of radius M. $H^1_c(\mathbb{R}^d) = \cup_{M\in\mathbb{N}} H^1_0(B(0,M))$ the set of compactly supported functions in $H^1(\mathbb{R}^d)$ and by $H^{1,+}_c(\mathbb{R}^d)$ the subset of those functions with nonnegative values.

In general, if $G \subset \mathbb{R}^k$, $\mathcal{D}(G)$ denotes the restriction to G of $\mathcal{D}(\mathbb{R}^k)$ functions u such that $\text{support}(u) \cap G$ is compact. Then, $\mathcal{D}^+(G)$ will denote the subset of non-negative elements of $\mathcal{D}(G)$.

For a given separable Banach space X we denote by $N^2_w(0,T,X)$ the space of the predictable X-valued processes (cf. [6, p. 94] or [18, p. 28] for example). This space is the space $L^2(]0,T[\times\Omega, X)$ for the product measure $dt \otimes dP$ on $\mathcal{P}_T$, the predictable σ-field (i.e. the σ-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $]s,t] \times A$ for any $A \in \mathcal{F}_s$).

If $X = L^2(\mathbb{R}^d)$, one gets that $N^2_w(0,T,L^2(\mathbb{R}^d)) \subset L^2(Q\times\Omega)$.

We denote $\mathcal{E}$ the set of non-negative convex function in $C^{2,1}(\mathbb{R})$ approximating the absolute-value function, such that $\eta(0) = 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ (respectively, -1) if $x > \delta$ (respectively, $x < -\delta$). Then, η'' has a compact support and η and η' are Lipschitz-continuous functions.

For convenience, denote by $\text{sgn}_0(x) = \frac{x}{|x|}$ if $x \neq 0$ and 0 otherwise; $F(a,b) = \text{sgn}_0(a-b)[\vec{\mathbf{f}}(a) - \vec{\mathbf{f}}(b)]$ and $F^\eta(a,b) = \int_b^a \eta'(\sigma - b)\vec{\mathbf{f}}'(\sigma)d\sigma$. Note, in particular, that F and F^η are Lipschitz-continuous functions. □

2. Entropy Formulation

Assume that for any positive ϵ, u_ϵ is the solution of the stochastic nonlinear parabolic problem

$$du - [\epsilon\Delta u + \text{div}(\vec{\mathbf{f}}(u))]dt = h(u)dw \quad \text{in } \Omega\times\mathbb{R}^d\times]0,T[\,, \tag{2.1}$$

for a smooth initial condition $u_0^\epsilon \in \mathcal{D}(\mathbb{R}^d)$. See Appendix A.2 for further information.

In order to propose an entropy formulation, let us analyze the viscous parabolic case. For this, consider φ in $\mathcal{D}^+(\bar{Q})$, k a real number and η in $\mathcal{E}$.

Since $\eta(u_\epsilon - k)\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$ a.s., it is possible to apply the Itô formula to the operator $\Psi(t,u_\epsilon) := \int_{\mathbb{R}^d} \eta(u_\epsilon - k)\varphi dx$ and thus we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \eta(u_\epsilon(T) - k)\varphi(T)dx \\
&\quad = \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)dx + \int_Q \eta(u_\epsilon - k)\partial_t\varphi dxdt \\
&\qquad - \epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi dxdt - \epsilon\int_Q \eta''(u_\epsilon - k)\varphi\nabla u_\epsilon\nabla u_\epsilon dxdt \\
&\qquad - \int_Q \eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla\varphi dxdt - \int_Q \eta''(u_\epsilon - k)\varphi\vec{\mathbf{f}}(u_\epsilon)\nabla u_\epsilon dxdt \\
&\qquad + \int_0^T\int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi dxdw(t) + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dxdt.
\end{aligned}$$

Since the support of η'' is compact, for any $i = 1, \ldots, d$, $\mathbb{R} \ni r \mapsto \eta''(r-k)f_i(r)$ is a bounded continuous function. Then, thanks to the chain-rule for Sobolev functions,

$$\begin{aligned}
&-\int_Q \eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla\varphi dxdt - \int_Q \eta''(u_\epsilon - k)\varphi\vec{\mathbf{f}}(u_\epsilon)\nabla u_\epsilon dxdt \\
&\quad = -\int_Q \eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla\varphi dxdt - \int_Q \varphi \operatorname{div}\left[\int_0^{u_\epsilon} \eta''(\sigma - k)\vec{\mathbf{f}}(\sigma)d\sigma\right] dxdt \\
&\quad = \int_Q \nabla\varphi \left[\int_0^{u_\epsilon} \eta''(\sigma - k)\vec{\mathbf{f}}(\sigma)d\sigma - \eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\right] dxdt \\
&\quad = -\int_Q \nabla\varphi \left[\int_k^{u_\epsilon} \eta'(\sigma - k)\vec{\mathbf{f}}'(\sigma)d\sigma\right] dxdt \\
&\quad = -\int_Q F^\eta(u_\epsilon, k)\nabla\varphi dxdt,
\end{aligned}$$

and

$$\begin{aligned}
0 \leq &\int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)dx + \int_Q \eta(u_\epsilon - k)\partial_t\varphi dxdt - \epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi dxdt \\
&- \int_Q F^\eta(u_\epsilon, k)\nabla\varphi dxdt + \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi dxdw(t) \\
&+ \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dxdt. \qquad (2.2)
\end{aligned}$$

Now let us assume that, as ϵ tends to 0, the approximate solutions u_ϵ converge in an appropriate sense to a function $u \in \mathcal{N}_w^2(0, T; L^2(\mathbb{R}^d))$ such that $\epsilon E \int_Q \mathbb{1}_A \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi dxdt$ tends to 0 (the convergence issue will be studied rigorously in Sec. 2). Then we may pass to the limit in the above inequality and obtain a family of entropy inequalities satisfied by the limit function u. This observation motivates the definition of entropy solution for the stochastic conservation law (1.1) we will give below.

For convenience, for any function u of $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d))$, any real k and any regular function η, denote dP-a.s. in Ω by $\mu_{\eta,k}$, the distribution in $\mathbb{R}^{d+1}$, defined by

$$\begin{aligned}
\varphi \mapsto \mu_{\eta,k}(\varphi) = &\int_{\mathbb{R}^d} \eta(u_0 - k)\varphi(0)dx + \int_Q \eta(u - k)\partial_t\varphi - F^\eta(u, k)\nabla\varphi dxdt \\
&+ \int_0^T \int_{\mathbb{R}^d} \eta'(u - k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u - k)\varphi dxdt.
\end{aligned}$$

Remark 2.1. Thanks to the Itô-integration by parts formula,

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(T,x) \int_0^T \eta'(u-k)h(u)dw(t)dx \\ = \int_0^T \int_{\mathbb{R}^d} \varphi_t(t,x) \int_0^t \eta'(u-k)h(u)dw(s)dxdt \\ + \int_0^T \int_{\mathbb{R}^d} \eta'(u-k)h(u)\varphi dxdw(t),\end{aligned}$$

and thus, dP-a.s. in Ω,

$$\begin{aligned}&\varphi \mapsto \mu_{\eta,k}(\varphi) \\ &= \int_{\mathbb{R}^d} \eta(u_0-k)\varphi(0)dx + \int_Q \eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi dxdt \\ &+ \int_{\mathbb{R}^d} \varphi(T,x) \int_0^T \eta'(u-k)h(u)dw(t)dx + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi dxdt \\ &- \int_0^T \int_{\mathbb{R}^d} \varphi_t(t,x) \int_0^t \eta'(u-k)h(u)dw(s)dxdt.\end{aligned}$$

From the preceding considerations, we are now naturally led to give the following definition.

Definition 2.2. A function u of $\mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))$ is an entropy solution of the stochastic conservation law (1.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$ if $u \in L^\infty(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$ and, for any $\varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$, any $k \in \mathbb{R}$ and any $\eta \in \mathcal{E}$

$$0 \le \mu_{\eta,k}(\varphi) \quad dP\text{-a.s.}$$

For technical reasons we also need to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we are able to deduce the existence of an entropy solution in the sense of Definition 2.2.

Definition 2.3. A function u in $\mathcal{N}_w^2[0,T,L^2(\mathbb{R}^d\times]0,1[)] \cap L^\infty[]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[)]$ is a (Young) measure-valued entropy solution of (1.1) with the initial data $u_0 \in L^2(\mathbb{R}^d)$ if for any $\eta \in \mathcal{E}$ and any $(k,\varphi) \in \mathbb{R}\times\mathcal{D}^+([0,T]\times\mathbb{R}^d)$,

$$0 \le \int_0^1 \mu_{\eta,k}(\varphi)d\alpha, \quad dP\text{-a.s.}$$

Note that in this definition the measure $\mu_{\eta,k}$ also depends on α because u does.

Remark 2.4. We will detail in Sec. 3 the proof of the existence of a measure-valued solution. It relies on the approximation of (1.1) by the viscous parabolic stochastic problems (2.1).

Since the sequence of solutions u_ϵ of (2.1) is bounded in $L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d))$ (cf. Proposition A.2), the compactness theorem of Prohorov (cf. Appendix A.3), ensures the existence of a Young measure limit $\mathbf{u}$.

Then, thanks to the *a priori* estimates and the compact support of the test-functions, one will be able to pass to the limit, in the sense of the Young measures in (2.2), and any limit-point of u_ϵ (where we keep the same notation u_ϵ for a subsequence) is an element of $L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[))$. In fact, according to Balder [1], for any non-negative Carathéodory function $\psi:]0,T[\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, we have

$$E\int_{Q\times]0,1[}\psi(\cdot,\mathbf{u})dxdtd\alpha\le\liminf_\epsilon E\int_Q\psi(\cdot,u_\epsilon)\,dx\,dt.$$

Now, if we choose $\psi(t,x,\omega,\lambda)=\beta(t)|\lambda|^2$ with $\beta\in L^1(0,T)$, $\beta\ge 0$, we find

$$\begin{aligned}\int_0^T\beta(t)E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2dxd\alpha dt&\le\liminf_\epsilon E\int_Q\beta(t)|u_\epsilon|^2\,dx\,dt\\&\le\liminf_\epsilon\|u_\epsilon\|^2_{L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))}\int_0^T\beta(t)\,dt\\&\le C\int_0^T\beta(t)\,dt\end{aligned}$$

as u_ϵ is bounded in $L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))$.

As, for any $\beta\in L^1(0,T)$, $\beta=\beta^+-\beta^-$ and, moreover, $t\mapsto E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,dx\,d\alpha$ is a measurable function, we get

$$\begin{aligned}\left|\int_0^T\beta(t)E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,dx\,d\alpha\,dt\right|&\le\liminf_\epsilon E\int_Q\beta(t)|u_\epsilon|^2\,dx\,dt\\&\le\liminf_\epsilon\|u_\epsilon\|^2_{L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))}\|\beta\|_{L^1(0,T)}\\&\le C\|\beta\|_{L^1(0,T)}\end{aligned}$$

which implies that $E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,dx\,d\alpha$ is an element of the dual of $L^1(0,T)$, i.e. an element of $L^\infty(0,T)$, hence $\mathbf{u}\in L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[))$.

Therefore it is natural to include in the definition the condition that an entropy (respectively, measure-valued entropy) solution belongs to $L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d))$ (respectively, $L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[))$).

Remark 2.5. If u is an entropy solution in the sense of Definition 2.2, then there exists a measurable set $\tilde\Omega\subset\Omega$ of full measure such that for any $\omega\in\tilde\Omega$, $0\le\mu_{\eta,k}(\varphi)$ for all $k\in\mathbb{Q}$, for all $\varphi\in D^+(\mathbb{R}^{d+1})$. Indeed, denote by $\mathcal{A}=\{a_M;\ M\in\mathbb{N}\}\subset\mathcal{D}^+(\mathbb{R}^{d+1})$ a countable dense sub-family of $\mathcal{D}^+(\mathbb{R}^{d+1})$ for the topology of $H^r(\mathbb{R}^{d+1})$ for arbitrary sufficiently large r.

From the definition it follows that there exists $\tilde\Omega_M\subset\Omega$ such that $P(\Omega\backslash\tilde\Omega_M)=0$ and, for any $\omega\in\tilde\Omega_M:0\le\mu_{\eta,k}(a_M)$ for all $k\in\mathbb{Q}$.

Now if $\tilde{\Omega} = \bigcap_M \tilde{\Omega}_M$, one gets that $P(\Omega \backslash \tilde{\Omega}) = 0$ and, for any $\omega \in \tilde{\Omega}$: $0 \leq \mu_{\eta,k}(\varphi)$ for all $k \in \mathbb{Q}$ and all $\varphi \in \mathcal{A}$.

Since by Remark 2.1 $\mu_{\eta,k}$ is a $H^r(\mathbb{R}^{d+1})$-continuous function, it follows that for any $\omega \in \tilde{\Omega}$: $0 \leq \mu_{\eta,k}(\varphi)$ for all $k \in \mathbb{Q}$, for all $\varphi \in D^+(\mathbb{R}^{d+1})$.

Thanks to Appendix A.1, $\mu_{\eta,k}$ is continuous with respect to k and the result holds by approximating any real number by a sequence of rational ones.

The same remark holds for a measure-valued entropy solution.

Let us also mention the following.

Remark 2.6. Any entropy solution is a.s. a weak solution, too.

Indeed, following Carrillo and Wittbold [2], dP-a.s, for any $\varphi \in \mathcal{D}^+([0,T[\times \mathbb{R}^d)$, we have

$$
\begin{aligned}
\mu_{\eta,k}(\varphi) = & \int_Q \{(u-k)\partial_t \varphi - [\vec{\mathbf{f}}(u) - \vec{\mathbf{f}}(k)] \cdot \nabla \varphi\}\, dx\, dt \\
& + \int_{\mathbb{R}^d} (u_0 - k)\varphi(0)\, dx & (:= I_1) \\
& + \int_Q [\eta(u-k) - u + k]\partial_t \varphi - [F^\eta(u,k) - \vec{\mathbf{f}}(u) + \vec{\mathbf{f}}(k)]\nabla \varphi dx\, dt \\
& + \int_{\mathbb{R}^d} [\eta(u_0 - k) - u_0 + k]\varphi(0)\, dx & (:= I_2) \\
& - \int_0^T \int_{\mathbb{R}^d} \varphi_t(t,x)\eta'(u-k)h(u)\, dx\, dw(t) & (:= I_3) \\
& + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi dx\, dt. & (:= I_4)
\end{aligned}
$$

Note that $I_1 = \int_Q \{u\partial_t \varphi - \vec{\mathbf{f}}(u) \cdot \nabla \varphi\}\, dx\, dt + \int_{\mathbb{R}^d} u_0 \varphi(0)\, dx$.

Since $\eta \in \mathcal{E}$, for $k < 0$, we get

$$
\begin{aligned}
& |F^\eta(u,k) - f(u) + f(k)| \\
& \quad = \left| \int_k^u \eta'(\sigma - k)\vec{\mathbf{f}}'(\sigma)\, d\sigma - f(u) - f(k) \right| = \left| \int_k^u [\eta'(\sigma - k) - 1]\vec{\mathbf{f}}'(\sigma)\, d\sigma \right| \\
& \quad \leq c(\vec{\mathbf{f}}') \left| \int_k^u [1 - \eta'(\sigma - k)]\, d\sigma \right| \leq c(\vec{\mathbf{f}}') \left| \int_0^{u-k} [1 - \eta'(\sigma)]\, d\sigma \right| \\
& \quad \leq c(\vec{\mathbf{f}}')[\delta + 2(u-k)^-] \leq c(\vec{\mathbf{f}}')[\delta + 2u^-],
\end{aligned}
$$

where δ is the parameter associated to η in the definition of the elements of $\mathcal{E}$.

Note that, as k goes to $-\infty$,

$$
\begin{aligned}
\eta(u-k) - (u-k) &= \int_0^{u-k} \eta'(r) - 1\, dr \to \int_0^\infty \eta'(r) - 1\, dr \\
&= \int_0^\delta \eta'(r) - 1\, dr = \eta(\delta) - \delta
\end{aligned}
$$

Therefore $\lim_{\delta \to 0} \lim_{k \to -\infty} I_2 = 0$. Similarly, $\lim_{k \to -\infty} I_4 = 0$.

Moreover, I_3 converges to $-\int_0^T \int_{\mathbb{R}^d} \varphi_t(t) \int_0^t h(u)dw(\sigma)\,dx\,dt$ when $k \to -\infty$. Thus, for any positive $\varphi \in \mathcal{D}([0,T[\times\mathbb{R}^d)$,

$$0 \leq \int_Q \left\{ \left(u - \int_0^t h(u)dw(\sigma) \right) \partial_t\varphi - \vec{\mathbf{f}}(u)\cdot\nabla\varphi \right\} dx\,dt + \int_{\mathbb{R}^d} u_0\varphi(0,\cdot)\,dx.$$

Since the opposite inequality can be proved by using $k-u$ instead of $u-k$ in I_1, passing to the limit when k goes to $+\infty$, we find that u is a solution in the sense of distributions.

Remark 2.7. The unique solution obtained in this paper satisfies the initial condition in the following sense: for any compact set $K \subset \mathbb{R}^d$,

$$\operatorname*{ess\,lim}_{t\to 0^+} E\int_K |u-u_0|\,dx = 0.$$

Indeed, by the existence proof, the solution u will be in $L^\infty\left(0,T,L^2(\Omega\times\mathbb{R}^d)\right)$. Therefore, following Otto [16] (see also Vallet [23]), if one considers any $k\in\mathbb{R}$ and any $\beta\in\mathcal{D}^+(\mathbb{R}^d)$, then, for any non-negative α in $H^1(0,T)$, one has that

$$\begin{aligned} 0 \leq& \int_0^T \left\{ \alpha' E\int_{\mathbb{R}^d} \eta(u-k)\beta\,dx + \alpha E\int_{\mathbb{R}^d} \frac{1}{2}h^2(u)\eta''(u-k)\beta - F^\eta(u,k)\cdot\nabla\beta\,dx \right\} dt \\ &+ E\int_0^T \alpha \int_{\mathbb{R}^d} \beta\eta'(u-k)h(u)\,dx\,dw(t) + \alpha(0)\int_{\mathbb{R}^d} \eta(u_0-k)\beta\,dx \\ =& \int_0^T [\alpha'(t)A_{k,\beta}(t) + \alpha(t)B_{k,\beta}(t)]\,dt + \alpha(0)C_{k,\beta}, \end{aligned}$$

where $A_{k,\beta}(t) = E\int_{\mathbb{R}^d}\eta(u-k)\beta\,dx$, $B_{k,\beta}(t) = E\int_{\mathbb{R}^d} h^2(u)\eta''(u-k)\beta - F^\eta(u,k)\cdot\nabla\beta\,dx$.

Therefore, $\mathbb{T} : \alpha\in\mathcal{D}^+(\mathbb{R}) \mapsto \int_0^T [\alpha'(t)A_{k,\beta}(t) + \alpha(t)B_{k,\beta}(t)]\,dt + \alpha(0)C_{k,\beta}$ is a positive Radon measure on $\mathbb{R}$. Its restriction to $]0,T[$, denoted by $\mathbb{T}_{]0,T[}$, is a positive bounded Radon measure on $]0,T[$ and

$$|\mathbb{T}_{]0,T[}| \leq E\mu_{\eta,k}(1\otimes\beta) = \int_0^T B_{k,\beta}(t)\,dt + C_{k,\beta} \leq C(\eta,\beta,\operatorname{supp}\beta,\|u\|_{L^2(Q\times\Omega)}).$$

In particular, $\psi : t\mapsto A_{k,\beta}(t) - \int_0^t B_{k,\beta}(s)\,ds$ is a nonincreasing function of bounded variation on $[0,T]$. Thus, $\psi(0^+) = \operatorname{ess\,lim}_{t\to 0^+}\psi(t)$ exists and

$$\psi(0^+) = \lim_{n\to\infty} n\int_0^{1/n} \psi(t)\,dt = \lim_{n\to\infty}\int_0^T \alpha_n'\psi(t)\,dt,$$

where $\alpha_n(t) = \min(nt,1)^+$.

Since $\lim_{t\to 0^+}\int_0^t B_{k,\beta}(s)\,ds = 0$, $A_{k,\beta}(0^+) = \operatorname{ess\,lim}_{t\to 0^+} A_{k,\beta}(t) = \psi(0^+)$ and

$$\begin{aligned} 0 \le A_{k,\beta}(0^+) &= \lim_{n\to\infty}\int_0^T (\alpha_n - 1)'\left[A_{k,\beta}(t) - \int_0^t B_{k,\beta}(s)\,ds\right] dt \\ &= \lim_{n\to\infty} -\int_0^T [(1-\alpha_n)' A_{k,\beta}(t) + B_{k,\beta}(t)(1-\alpha_n)]\,dt, \\ &= \lim_{n\to\infty}\big[-\mu_{\eta,k}[(1-\alpha_n)\otimes\beta]\big] + C_{k,\beta} \le \int_{\mathbb{R}^d}\eta(u_0 - k)\beta\,dx. \end{aligned}$$

Let us fix $K \subset \mathbb{R}^d$ a compact set and denote by $L^2_K(\mathbb{R}^d)$ the Hilbert-subspace of $L^2(\mathbb{R}^d)$ of functions with support in K.

Thanks to the hypothesis on u, uniformly with respect to t, $\beta \mapsto A_{k,\beta}(t)$ is a continuous linear function on $L^2_K(\mathbb{R}^d)$, and thus a density argument leads to the existence, for any real k and any non-negative β in $L^2_K(\mathbb{R}^d)$, of $\operatorname{ess\,lim}_{t\to 0^+} A_{k,\beta}(t) = A_{k,\beta}(0^+)$ with, moreover, $A_{k,\beta}(0^+) \le \int_{\mathbb{R}^d}\eta(u_0 - k)\beta\,dx$.

In order to keep essential limits, consider k in $\mathbb{Q}$. Then, if $w_n = \sum_{i=0}^n k_i 1_{B_i}$ is a simple function with k_i in $\mathbb{Q}$, one gets that

$$\begin{aligned} A_{w_n,\beta}(t) &= E\int_{\mathbb{R}^d}\eta(u(t) - w_n)\beta\,dx = \sum_{i=0}^n E\int_{\mathbb{R}^d}\eta(u(t)-k_i)\beta 1_{B_i}\,dx \\ &= \sum_{i=0}^n A_{k_i,\beta}1_{B_i}(t), \end{aligned}$$

and $\operatorname{ess\,lim}_{t\to 0^+} A_{w_n,\beta}(t)$ exists with, moreover, $A_{w_n,\beta}(0^+) \le \int_{\mathbb{R}^d}\eta(u_0 - w_n)\beta\,dx$, for any non-negative β in $L^2_K(\mathbb{R}^d)$ and any $\mathbb{Q}$-valued simple function w_n.

As any w of $L^2(\mathbb{R}^d)$ is a limit in $L^2(\mathbb{R}^d)$ of a sequence of such simple functions and since for w and $\hat{w}$ in $L^2(\mathbb{R}^d)$, $|A_{w,\beta}(t) - A_{\hat{w},\beta}(t)| \le \|w - \hat{w}\|_{L^2(\mathbb{R}^d)}\|\beta\|_{L^2(\mathbb{R}^d)}$, independently of t, the same argument of density leads to: $\operatorname{ess\,lim}_{t\to 0^+} A_{w,\beta}(t)$ exists with, moreover, $A_{w,\beta}(0^+) \le \int_{\mathbb{R}^d}\eta(u_0 - w)\beta\,dx$, for any non-negative β in $L^2_K(\mathbb{R}^d)$ and any w in $L^2(\mathbb{R}^d)$.

Now, for $w = u_0$ and $\beta = 1_K$ this leads to: $\operatorname{ess\,lim}_{t\to 0^+} E\displaystyle\int_K \eta(u(t) - u_0)\,dx = 0$.

Since it is possible to approximate the absolute value function from below by a nondecreasing sequence of functions in $\mathcal{E}$, the theorem of Dini assures us the uniform convergence of the sequence. Thus, $\operatorname{ess\,lim}_{t\to 0^+} E\int_K |u(t) - u_0|\,dx = 0$.

Remark 2.8. Replacing $\int_{\mathbb{R}^d}\cdots dx$ by $\int_{\mathbb{R}^d\times]0,1[}\cdots dx\,d\alpha$ in the preceding arguments, one can prove in the same way that $\operatorname{ess\,lim}_{t\to 0^+} E\int_{K\times]0,1[}|u(t) - u_0|\,dx\,d\alpha = 0$ for a measure-valued entropy solution.

The main result of this paper is the following.

Theorem 2.9. *Under assumptions $H_1 - H_2 - H_3$ there exists a unique measure-valued entropy solution in the sense of Definition* 2.3.

Moreover, this solution is the unique entropy solution of (1.1), *for any initial condition* u_0 *in* $L^2(\mathbb{R}^d)$, *in the sense of Definition* 2.2.

If u_1, u_2 *are entropy solutions of* (1.1) *corresponding to initial data* $u_{1,0}, u_{2,0}$ *in* $L^2(\mathbb{R}^d)$, *respectively, then, for any* $K > 0$ *and any* t,

$$E\int_{B(0,K-\omega t)} |u_1 - u_2|\,dx \leq \int_{B(0,K)} |u_{1,0} - u_{2,0}|\,dx.$$

Remark 2.10. Following [24, Sec. 6.1], if $0 \leq u_0 \leq 1$ and if $\operatorname{supp} h \subset [0,1]$, then $0 \leq u \leq 1$.

3. Existence of a Solution

The aim of this section is to prove the following.

Theorem 3.1. *Under assumptions* $H_1 - H_2 - H_3$ *there exists a measure-valued entropy solution in the sense of Definition* 2.3.

The technique is based on the notion of narrow convergence of Young measures (or entropy processes) (cf. Appendix A.3). Then, thanks to the uniqueness result of the next section, we will be able to prove that the measure-valued solution is an entropy solution in the sense of Definition 2.2 and that the sequence of approximation proposed to prove the existence of the solution converges in $L^p(]0,T[\times\Omega, L^p_{\mathrm{loc}}(\mathbb{R}^d))$ for any $1 \leq p < 2$.

Thanks to Appendix A.2, for any positive ϵ, there exists a unique weak solution u_ϵ of the stochastic viscous parabolic equation:

$$\partial_t\left[u - \int_0^t h(u)\,dw(s)\right] - \epsilon\Delta u - \operatorname{div}(\vec{\mathbf{f}}(u)) = 0,$$

associated with a regular initial condition u_0^ϵ.

More precisely, there exists u_ϵ in $\mathcal{N}_w^2(0,T;H^1(\mathbb{R}^d)) \cap C([0,T],L^2(\Omega\times\mathbb{R}^d))$ with moreover Δu_ϵ and $\partial_t[u_\epsilon - \mathcal{K}]$ in $L^2(]0,T[\times\Omega;L^2(\mathbb{R}^d))$ where $\mathcal{K} = \int_0^t h(u_\epsilon)dw(s)$, $u_\epsilon(t=0) = u_0^\epsilon$ and, a.s. in Ω, a.e. in $]0,T[$, for any v in $H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\left[\partial_t\left[u_\epsilon - \int_0^t h(u_\epsilon)\,dw(s)\right]v + \epsilon\nabla u_\epsilon\cdot\nabla v + \vec{\mathbf{f}}(u_\epsilon)\cdot\nabla v\right]dx = 0. \qquad (3.1)$$

In the same section the following uniform *a priori* estimate on the approximate solutions is proved.

Lemma 3.2. *Assume that* $(u_\epsilon^0)_\epsilon$ *is bounded in* $L^2(\mathbb{R}^d)$. *Then there exists a positive constant* C *such that*

$$\|u_\epsilon\|^2_{L^\infty[0,T;L^2(\Omega\times\mathbb{R}^d)]} + \epsilon\|u_\epsilon\|^2_{L^2[]0,T[\times\Omega;H_0^1(\mathbb{R}^d)]} \leq C.$$

If, for some $p \geq 1$, $u_0^\epsilon \in L^{2p}(\mathbb{R}^d)$, *then* $u_\epsilon \in L^\infty(0,T,L^{2p}(\Omega\times\mathbb{R}^d))$.

Moreover, thanks to Sec. 2, for any $\varphi \in \mathcal{D}^+(\bar{Q})$, any $k \in \mathbb{R}$ and any $\eta \in \mathcal{E}$,

$$0 \leq \mu^\epsilon_{\eta,k}(\varphi) - \epsilon \int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi dx\, dt, \quad dP\text{-a.s.}$$

Then, for any dP-measurable set A, one gets that

$$\begin{aligned} 0 \leq E&\left[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi dx\, dw(t)\right] \\ &- \epsilon E\left[1_A \int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi dx\, dt\right] + E\left[1_A \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\, dx\right] \\ &+ E\left[1_A \int_Q \eta(u_\epsilon - k)\partial_t\varphi - F^\eta(u_\epsilon, k)\nabla\varphi + \frac{1}{2}h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dx\, dt\right]. \end{aligned} \tag{3.2}$$

Since u_ϵ is a bounded sequence in $\mathcal{N}^2_w(0, T, L^2(\mathbb{R}^d))$ and thanks to the compact support of φ in $\mathbb{R}^d$, the associated Young measure sequence u_ϵ converges (up to a subsequence still indexed in the same way) to an "entropy process" denoted by $\mathbf{u}$ (see (A.5) in the Appendix).

By assumptions on η, all the integrands in (3.2) are uniformly integrable integrands and passing to the limit is possible in all the integrals.

Thanks to Appendix A.3 one is also able to pass to the limit in the first term of (3.2) and the *a priori* estimates yield that the second one tends to 0 with ϵ. Therefore at the limit one gets

$$\begin{aligned} 0 \leq E&\left[1_A \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mathbf{u}(\cdot,\alpha) - k)h(\mathbf{u}(\cdot,\alpha))\varphi d\alpha\, dx\, dw(t)\right] \\ &+ \frac{1}{2}E\left[1_A \int_Q \int_0^1 h^2(\mathbf{u}(\cdot,\alpha))\eta''(\mathbf{u}(\cdot,\alpha) - k)\varphi d\alpha\, dx\, dt\right] \\ &+ E\left[1_A \int_{\mathbb{R}^d} \eta(u_0 - k)\varphi(0)\, dx\right] \\ &+ E\left[1_A \int_Q \int_0^1 [\eta(\mathbf{u}(\cdot,\alpha) - k)\partial_t\varphi - F^\eta(\mathbf{u}(\cdot,\alpha), k)\nabla\varphi]\, d\alpha\, dx\, dt\right]. \end{aligned}$$

A separability argument for the norm of $H^1(Q)$ yields the existence of a Young measure solution.

4. Uniqueness

The aim of this section is to prove the following.

Theorem 4.1. *The solution given by Theorem* 3.1 *is the unique measure-valued entropy solution in the sense of Definition* 2.3. *Moreover, it is the unique entropy solution in the sense of Definition* 2.2.

Remark 4.2. We will also prove for entropy solutions a result of stability (contraction principle) in L^1 (see Proposition 4.5 below).

4.1. *Kato inequality*

The aim of this section is to prove the following interior Kato inequality:

Proposition 4.3. *Let u_1, u_2 be Young measure-valued entropy solution to* (1.1) *with initial data $u_{1,0}, u_{2,0} \in L^2(\mathbb{R}^d)$, respectively. Then, for any non-negative $H^1(\mathbb{R}^{d+1})$-function φ with compact support, it holds*

$$0 \leq \int_{\mathbb{R}^d} |u_{1,0} - u_{2,0}|\varphi(0)\,dx + E\int_{Q\times]0,1[^2} |u_1(t,x,\alpha) - u_2(t,x,\beta)|\partial_t\varphi\,dx\,dt d\alpha d\beta$$
$$- E\int_{Q\times]0,1[^2} F(u_1(t,x,\alpha), u_2(t,x,\beta)) \cdot \nabla\varphi\,dx\,dt d\alpha d\beta.$$

Let us denote by $\mathbf{u}$ the Young measure entropy solution from the previous section (a limit point of (u_ϵ)) and $\hat{\mathbf{u}}$ any other admissible Young measure-valued entropy solution associated to two initial conditions u_0 and $\hat{u}_0$ in $L^2(\mathbb{R}^d)$, respectively.

In a first step we will prove the local Kato inequality for $u_1 = \hat{\mathbf{u}}$ and $u_2 = \mathbf{u}$. In a second step (see Sec. 4.2), exploiting the finite propagation speed property for conservation laws with Lipschitz-continuous flux function, we will deduce from the local Kato inequality a global one and, in particular, obtain a local L^1-contraction principle (see Proposition 4.5 below).

As a consequence (choosing $u_{1,0} = \hat{u}_0 = u_0 = u_{2,0}$) we deduce that $\mathbf{u} = \hat{\mathbf{u}}$ and thus any Young measure-valued solution is obtained as the limit of solutions u_ϵ of viscous parabolic approximations to (1.1).

Then it follows immediately that, in fact, the local Kato inequality also holds for any arbitrary pair of measure-valued entropy solutions.

In order to prove the local Kato inequality for $\hat{\mathbf{u}}$, $\mathbf{u}$, consider φ in $\mathcal{D}^+([0,T]\times\mathbb{R}^d)$, $K \subset \mathbb{R}^d$ a compact set such that supp $\varphi(t,\cdot) \subset K$ and denote by $G(t,x,s,y) = \varphi(s,y)\rho_m(x-y)\rho_n(t-s)$ where ρ_m and ρ_n denote the usual mollifier sequences in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, with supp$\rho_n \subset [-\frac{2}{n}, 0]$.

Denote also by ρ_l a mollifier sequence in $\mathbb{R}$ and for convenience set $p = (t,x,\alpha)$. Since $\hat{\mathbf{u}} = \hat{\mathbf{u}}(p)$ is a Young measure solution, by considering the test-function G, multiplying the entropy formulation by $\mathcal{B}_k^l := \rho_l(u_\epsilon(s,y) - k)$ and integrating k over $\mathbb{R}$, we get, on the one hand, that

$$0 \leq E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x) - k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,dx\mathcal{B}_k^l\,dk\,dy\,ds$$
$$+ E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(\hat{\mathbf{u}}(p) - k)\partial_t\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,dp\mathcal{B}_k^l\,dk\,dy\,ds$$
$$+ E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(\hat{\mathbf{u}}(p) - k)\varphi(s,y)\partial_t\rho_n(t-s)\rho_m(x-y)\,dp\mathcal{B}_k^l\,dk\,dy\,ds$$
$$- E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}}(p), k)\nabla_x\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,dp\mathcal{B}_k^l\,dk\,dy\,ds$$

$$- E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,dp\mathcal{B}_k^l\,dk\,dy\,ds$$

$$+ \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y)\rho_n(t-s)$$

$$\times\varphi(s,y)\,dp\mathcal{B}_k^l\,dk\,dy\,ds$$

$$+ E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,d\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)$$

$$\times\,dx\,dw(t)\mathcal{B}_k^l\,dk\,dy\,ds$$

$$=: I_1+I_2+I_3+I_4+I_5+I_6+I_7.$$

On the other hand, if one denotes $\mathcal{A}_k^l = \rho_l(\hat{\mathbf{u}}(p)-k)$, since u_ϵ is a viscous solution, one gets that

$$0 \le E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(u_0^\epsilon(y)-k)\varphi(0,y)\rho_n(t)\rho_m(x-y)\,dy\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$+ E\int_Q\int_{\mathbb{R}}\int_Q\eta(u_\epsilon(s,y)-k)\partial_s\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,dy\,ds\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$+ E\int_Q\int_{\mathbb{R}}\int_Q\eta(u_\epsilon(s,y)-k)\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\,dy\,ds\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$- \epsilon E\int_Q\int_{\mathbb{R}}\int_Q\eta'(u_\epsilon(s,y)-k)\nabla_y u_\epsilon(s,y)\nabla_y\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,dy\,ds$$

$$\times\int_0^1 \mathcal{A}_k^l\,dk\,dp - \epsilon E\int_Q\int_{\mathbb{R}}\int_Q\eta'(u_\epsilon(s,y)-k)\nabla_y u_\epsilon(s,y)$$

$$\times\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,dy\,ds\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$- E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,dy\,ds\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$- E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,dy\,ds\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$+ \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\eta''(u_\epsilon(s,y)-k)\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,dy\,ds$$

$$\times\int_0^1 \mathcal{A}_k^l\,dk\,dp + E\int_Q\int_{\mathbb{R}}\int_Q\eta'(u_\epsilon(s,y)-k)h(u_\epsilon(s,y))\varphi(s,y)$$

$$\times\rho_m(x-y)\rho_n(t-s)\,dy\,dw(s)\int_0^1 \mathcal{A}_k^l\,dk\,dp$$

$$=: J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8+J_9.$$

Summing up the preceding two inequalities, our aim is now to pass to the limit in the following order: 1. $n \to \infty$ (time), 2. $l \to \infty$, 3. $\eta \to |\cdot|$, 4. $\epsilon \to 0$, 5. $m \to \infty$ (space).

In the following, as a uniform approximation of the absolute value function, we choose $\eta = \eta_\delta \in \mathcal{E}$ with $\eta'_\delta(r) = 1$ for $r > \delta$, $= \sin(\frac{\pi}{2\delta} r)$ if $|r| \leq \delta$ and $= -1$ for $r < -\delta$.

First let us consider

(1)

$$\begin{aligned} I_1 + J_1 = E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} & \eta(\hat{u}_0(x) - k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,dx \\ & \times \rho_l(u_\epsilon(s,y) - k)\,dk\,dy\,ds \\ & + E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta(u_0^\epsilon(y) - k)\rho_n(t)\varphi(0,y)\rho_m(x-y)\,dy \\ & \times \int_0^1 \rho_l(\mathbf{\hat{u}}(t,x,\alpha) - k)\,dk\,dp \\ = E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} & \eta(\hat{u}_0(x) - k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,dx \\ & \times \rho_l(u_\epsilon(s,y) - k)\,dk\,dy\,ds \end{aligned}$$

as supp $\rho_n \subset [-\frac{2}{n}, 0]$.

Therefore,

$$\begin{aligned} I_1 + J_1 &\xrightarrow[n\to\infty]{} E \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta(\hat{u}_0(x) - k)\varphi(0,y)\rho_m(x-y)\,dx\,\rho_l(u_0^\epsilon(y) - k)\,dk\,dy \\ &\xrightarrow[l\to\infty]{} E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta(\hat{u}_0(x) - u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,dx\,dy \\ &\xrightarrow[\eta\to|.|]{} E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{u}_0(x) - u_0^\epsilon(y)|\,\varphi(0,y)\rho_m(x-y)\,dx\,dy \\ &\xrightarrow[\epsilon\to 0]{} E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{u}_0(x) - u_0(y)|\,\varphi(0,y)\rho_m(x-y)\,dx\,dy \\ &\xrightarrow[m\to\infty]{} E \int_{\mathbb{R}^d} |\hat{u}_0(x) - u_0(x)|\,\varphi(0,x)\,dx. \end{aligned}$$

For the convenience of the reader let us justify all passages to the limit in detail. As to the passage to the limit with $n \to \infty$, note that, by a simple change of variables,

$$\begin{aligned} \mathcal{A}_1 := E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} & \eta(\hat{u}_0(x) - k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,dx \\ & \times \rho_l(u_\epsilon(s,y) - k)\,dk\,dy\,ds \\ & - E \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta(\hat{u}_0(x) - k)\varphi(0,y)\rho_m(x-y)\,dx\,\rho_l(u_0^\epsilon(y) - k)\,dk\,dy \end{aligned}$$

$$
\begin{aligned}
&= E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)\varphi(s,y)\rho_n(-s)\\
&\quad\times \rho_m(x-y)\,dx\,\rho_l(k)\,dk\,dy\,ds\\
&\quad - E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_0^\epsilon(y)+k)\varphi(0,y)\rho_m(x-y)\,dx\,\rho_l(k)\,dk\,dy\\
&= E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)[\varphi(s,y)-\varphi(0,y)]\rho_n(-s)\\
&\quad\times \rho_m(x-y)\,dx\,\rho_l(k)\,dk\,dy\,ds\\
&\quad + E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} [\eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)-\eta(\hat{u}_0(x)-u_0^\epsilon(y)+k)]\varphi(0,y)\\
&\quad\times \rho_n(-s)\rho_m(x-y)\rho_l(k)\,dx\,dk\,dy\,ds,
\end{aligned}
$$

and thus, setting $\mathcal{G}_n^m := \rho_n(-s)\rho_m(x-y)$,

$$
\begin{aligned}
|\mathcal{A}_1| &\le \|\varphi_t\|_\infty E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} 1_K(y)\eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)s\mathcal{G}_n^m\,dx\,\rho_l(k)\,dk\,dy\,ds\\
&\quad + \|\eta'\|_\infty E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} |u_\epsilon(s,y)-u_0^\epsilon(y)|\varphi(0,y)\mathcal{G}_n^m\,dx\,\rho_l(k)\,dk\,dy\,ds\\
&\le \frac{\|\varphi_t\|_\infty\|\eta'\|_\infty}{n} E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} 1_K(y)[|\hat{u}_0(x)-u_\epsilon(s,y)|+|k|]\mathcal{G}_n^m\,dx\,\rho_l(k)\,dk\,dy\,ds\\
&\quad + \|\eta'\|_\infty E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} |u_\epsilon(s,y)-u_0^\epsilon(y)|\varphi(0,y)\mathcal{G}_n^m\,dx\,\rho_l(k)\,dk\,dy\,ds,\\
|\mathcal{A}_1| &\le \frac{\|\varphi_t\|_\infty\|\eta'\|_\infty}{n} E\int_Q\int_{\mathbb{R}^d} 1_K(y)[|\hat{u}_0(x)-u_\epsilon(s,y)|+1]\mathcal{G}_n^m\,dx\,dy\,ds\\
&\quad + \|\varphi\|_\infty\|\eta'\|_\infty E\int_0^T\int_K |u_\epsilon(s,y)-u_0^\epsilon(y)|\rho_n(-s)\,dy\,ds\\
&\le \frac{\|\varphi_t\|_\infty\|\eta'\|_\infty}{n}\\
&\quad\times\left[\|\hat{u}_0(x)\|_{L^1(\mathbb{R}^d)} + E\int_K\int_0^T |u_\epsilon(s,y)|\rho_n(-s)\,ds\,dy + \text{Meas}(K)\right]\\
&\quad + \|\varphi\|_\infty\|\eta'\|_\infty E\int_0^T\int_K |u_\epsilon(s,y)-u_0^\epsilon(y)|\rho_n(-s)\,dy\,ds \xrightarrow[n\to\infty]{} 0
\end{aligned}
$$

as $u_\epsilon \in C([0,T], L^2(\Omega\times\mathbb{R}^d))$ with $u_\epsilon(0,\cdot) = u_0^\epsilon$.

Next, let us consider the passage to the limit with $l \to \infty$.

$$
\begin{aligned}
\mathcal{A}_2 &:= E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-k)\varphi(0,y)\rho_m(x-y)\,dx\,\rho_l(u_0^\epsilon(y)-k)\,dk\,dy\\
&\quad - E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,dx\,dy
\end{aligned}
$$

$$= E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d}[\eta(\hat{u}_0(x)-k)-\eta(\hat{u}_0(x)-u_0^\epsilon(y))]\varphi(0,y)$$
$$\times\rho_m(x-y)\,dx\,\rho_l(u_0^\epsilon(y)-k)\,dk\,dy.$$

$$|\mathcal{A}_2| \le \|\eta'\|_\infty E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|u_0^\epsilon(y))-k|\varphi(0,y)\rho_m(x-y)\,dx\,\rho_l(u_0^\epsilon(y)-k)\,dk\,dy$$
$$\le \frac{\|\eta'\|_\infty}{l}\|\varphi\|_\infty \mathrm{Meas}(K) \xrightarrow[l\to\infty]{} 0.$$

As to the passage to the limit with $\eta=\eta_\delta\to|\cdot|$, note that

$$\mathcal{A}_3 := E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta(\hat{u}_0(x)-u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,dx\,dy$$
$$-E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{u}_0(x)-u_0^\epsilon(y)|\,\varphi(0,y)\rho_m(x-y)\,dx\,dy$$
$$= E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\eta(\hat{u}_0(x)-u_0^\epsilon(y))-|\hat{u}_0(x)-u_0^\epsilon(y)|]\,\varphi(0,y)\rho_m(x-y)\,dx\,dy.$$

As $|\eta(r)-|r||\le\delta$ for any $r\in\mathbb{R}$, we have

$$|\mathcal{A}_3|\le\delta\|\varphi\|_\infty\mathrm{Meas}(K)\to_{\delta\to 0} 0.$$

Finally, consider the passage to the limit with $\epsilon\to 0$. As

$$\mathcal{A}_4 := E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{u}_0(x)-u_0^\epsilon(y)|\,\varphi(0,y)\rho_m(x-y)\,dx\,dy$$
$$-E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{u}_0(x)-u_0(y)|\,\varphi(0,y)\rho_m(x-y)\,dx\,dy$$
$$= E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[|\hat{u}_0(x)-u_0^\epsilon(y)|-|\hat{u}_0(x)-u_0(y)|]\,\varphi(0,y)\rho_m(x-y)\,dx\,dy,$$

by the reverse triangle inequality, we have

$$|\mathcal{A}_4|\le E\int_K|u_0(y)-u_0^\epsilon(y)|\,dy$$

which tends to 0 as $u_0^\epsilon\to u_0$ in $L^2(\mathbb{R}^d)$ as $\epsilon\to 0$.

(2) As φ is a function of variables (s,y),

$$I_2+J_2 = E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta(\hat{\mathbf{u}}(p)-k)\partial_t\varphi(s,y)\rho_n(t-s)$$
$$\times\rho_m(x-y)\,dp\,\rho_l(u_\epsilon(s,y)-k)\,dk\,dy\,ds$$
$$+E\int_Q\int_{\mathbb{R}}\int_Q\eta(u_\epsilon(s,y)-k)\partial_s\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,dy\,ds$$
$$\times\int_0^1\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,dk\,dp$$

$$= E\int_Q\int_{\mathbb{R}}\int_Q \eta(u_\epsilon(s,y)-k)\partial_s\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,dy\,ds$$

$$\times\int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,dk\,dp$$

$$\xrightarrow[n\to\infty]{} E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(u_\epsilon(s,y)-k)\partial_s\varphi(s,y)\rho_m(x-y)\,dy$$

$$\times\int_0^1 \rho_l(\hat{\mathbf{u}}(s,x,\alpha)-k)\,dk\,dx\,ds\,d\alpha$$

$$\xrightarrow[l\to\infty]{} E\int_Q\int_{\mathbb{R}^d}\int_0^1 \eta(u_\epsilon(s,y)-\hat{\mathbf{u}}(s,x,\alpha))\partial_s\varphi(s,y)$$

$$\times\,\rho_m(x-y)\,dy\,dx\,ds\,d\alpha$$

$$\xrightarrow[\eta\to|\cdot|]{} E\int_Q\int_{\mathbb{R}^d}\int_0^1 |u_\epsilon(s,y)-\hat{\mathbf{u}}(s,x,\alpha)|\,\partial_s\varphi(s,y)$$

$$\times\,\rho_m(x-y)\,dy\,dx\,ds\,d\alpha$$

$$\xrightarrow[\epsilon\to 0]{} E\int_Q\int_{\mathbb{R}^d}\int_0^1\int_0^1 |\mathbf{u}(s,y,\beta)-\hat{\mathbf{u}}(s,x,\alpha)|\,\partial_s\varphi(s,y)$$

$$\times\,\rho_m(x-y)\,dy\,dx\,ds\,d\alpha\,d\beta$$

$$\xrightarrow[m\to\infty]{} E\int_Q\int_0^1\int_0^1 |\mathbf{u}(s,y,\beta)-\hat{\mathbf{u}}(s,y,\alpha)|\,\partial_s\varphi(s,y)\,dy\,ds\,d\alpha\,d\beta.$$

The passages to the limit is similar to the previous one, by using moreover the properties of convolution in $L^1(0,T,L^1(\Omega\times\mathbb{R}^d\times(0,1))$ and the absolute continuity of the integral on the one hand, and the limit in the sense of Young measures with the Carathéodory-function $F(k,s,y) = \int_{\mathbb{R}^d}\int_0^1 |k-\hat{\mathbf{u}}(s,x,\alpha)|\,\partial_s\varphi(s,y)\rho_m(x-y)\,dx\,d\alpha$ on the other hand.

(3)

$$I_3+J_3 = E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(\hat{\mathbf{u}}(p)-k)\varphi(s,y)\partial_t\rho_n(t-s)$$

$$\times\,\rho_m(x-y)\,dp\,\rho_l(\underbrace{u_\epsilon(s,y)-k}_{=\tau})\,dk\,dy\,ds$$

$$+\,E\int_Q\int_{\mathbb{R}}\int_Q \eta(u_\epsilon(s,y)-k)\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\,dy\,ds$$

$$\times\int_0^1 \rho_l(\underbrace{\hat{\mathbf{u}}(p)-k}_{=-\sigma})\,dk\,dp$$

$$= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(\hat{\mathbf{u}}(p)-u_\epsilon(s,y)+\tau)\varphi(s,y)\partial_t\rho_n(t-s)$$

$$\times\,\rho_m(x-y)\,dp\,\rho_l(\tau)\,d\tau\,dy\,ds$$

$$+ E \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta(u_\epsilon(s,y) - \hat{\mathbf{u}}(p) - \sigma)\varphi(s,y)\partial_s \rho_n(t-s)$$
$$\times \rho_m(x-y)\, dy\, ds \rho_l(-\sigma)\, d\sigma\, dp = 0$$

since η and ρ_l are even functions.

(4)

$$J_4 + J_5 = -\epsilon E \int_Q \int_{\mathbb{R}} \int_Q \eta'(u_\epsilon(s,y) - k)\nabla_y u_\epsilon(s,y)\nabla_y \varphi(s,y)\rho_n(t-s)$$
$$\times \rho_m(x-y)\, dy\, ds \int_0^1 \mathcal{A}_k^l\, dk\, dp$$
$$- \epsilon E \int_Q \int_{\mathbb{R}} \int_Q \eta'(u_\epsilon(s,y) - k)\nabla_y u_\epsilon(s,y)\nabla_y \rho_m(x-y)\rho_n(t-s)$$
$$\times \varphi(s,y)\, dy\, ds \int_0^1 \mathcal{A}_k^l\, dk\, dp$$
$$\xrightarrow[n\to\infty]{} -\epsilon E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 \eta'(u_\epsilon(t,y) - k)\nabla_y u_\epsilon(t,y)$$
$$\times \nabla_y[\varphi \rho_m(x-y)]\mathcal{A}_k^l\, dy\, dk\, dp$$
$$\xrightarrow[l\to\infty]{} -\epsilon E \int_Q \int_{\mathbb{R}^d} \int_0^1 \eta'(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha))\nabla_y u_\epsilon(t,y)$$
$$\times \nabla_y[\varphi(t,y)\rho_m(x-y)]\, dy\, dp$$
$$\xrightarrow[\eta\to|\cdot|]{} -\epsilon E \int_Q \int_{\mathbb{R}^d} \int_0^1 \operatorname{sgn}_0\left(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha)\right)\nabla_y u_\epsilon(t,y)$$
$$\times \nabla_y[\varphi(t,y)\rho_m(x-y)]\, dy\, dp \xrightarrow[\epsilon\to 0]{} 0,$$

where

$$\mathcal{A}_k^l = \rho_l(\hat{\mathbf{u}}(t,x,\alpha) - k).$$

The passages to the limit with n and $l \to \infty$ follow by classical arguments for convolutions in the deterministic setting. The passage to the limit with $\eta \to |\cdot|$ follows from point-wise convergence and with Lebesgue's dominated convergence theorem. The convergence with $\epsilon \to 0$ follows from the *a priori* estimate Lemma 3.2 which implies that $\epsilon \nabla u_\epsilon$ converges to 0 in $L^2(]0,T[\times\Omega, L^2(\mathbb{R}^d))$.

(5) Since φ is a function of variables (s,y), denoting $\mathcal{A}_k^l = \rho_l(\hat{\mathbf{u}}(t,x,\alpha) - k)$

$$I_4 + J_6 = -E \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 F^\eta(\hat{\mathbf{u}}(p), k)\nabla_x \varphi(s,y)\rho_m(x-y)$$
$$\times \rho_n(t-s)\, dp \rho_l(u_\epsilon(s,y) - k)\, dk\, dy\, ds$$
$$- E \int_Q \int_{\mathbb{R}} \int_Q F^\eta(u_\epsilon(s,y), k)\nabla_y \varphi(s,y)\rho_m(x-y)\rho_n(t-s)\, dy\, ds$$
$$\times \int_0^1 \mathcal{A}_k^l\, dk\, dp$$

$$= -E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,dy\,ds$$
$$\times\int_0^1 \mathcal{A}_k^l\,dk\,dp,$$

and

$$I_4+J_6 \xrightarrow[n\to\infty]{} -E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(u_\epsilon(t,y),k)\nabla_y\varphi(t,y)\rho_m(x-y)\mathcal{A}_k^l\,dy\,dk\,dp$$
$$\xrightarrow[l\to\infty]{} -E\int_{\mathbb{R}^d}\int_Q\int_0^1 F^\eta(u_\epsilon(t,y),\hat{\mathbf{u}}(t,x,\alpha))\nabla_y\varphi(t,y)\rho_m(x-y)\,dy\,dp$$
$$\xrightarrow[\eta\to|\cdot|]{} -E\int_{\mathbb{R}^d}\int_Q\int_0^1 F(u_\epsilon(t,y),\hat{\mathbf{u}}(t,x,\alpha))\nabla_y\varphi(t,y)\rho_m(x-y)\,dy\,dp$$
$$\xrightarrow[\epsilon\to 0]{} -E\int_{\mathbb{R}^d}\int_Q\int_0^1\int_0^1 F(\mathbf{u}(t,y,\beta),\hat{\mathbf{u}}(t,x,\alpha))\nabla\varphi(t,y)\rho_m(x-y)\,dy\,dp\,d\beta$$
$$\xrightarrow[m\to\infty]{} -E\int_Q\int_0^1\int_0^1 F(\mathbf{u}(t,x,\beta),\hat{\mathbf{u}}(t,x,\alpha))\nabla\varphi(t,x)\,dp\,d\beta.$$

The passages to the limit is similar to the previous ones by noticing that $F^\eta(\cdot,k)$ is a Lipschitz-continuous function with Lipschitz-constant $\|\vec{f}'\|_\infty$ and that $G(k,t,y)=\int_{\mathbb{R}^d}\int_0^1 F(k,\hat{\mathbf{u}}(t,x,\alpha))\nabla_y\varphi(t,y)\rho_m(x-y)\,dx\,d\alpha$ is a Carathéodory-function.

(6) Now let us consider I_5+J_7: As $\nabla_y\rho_m(x-y)=-\nabla_x\rho_m(x-y)$, using similar arguments as before (symmetry of F, i.e. $F(r,s)=F(s,r)$, the fact that for $\eta=\eta_\delta$: $|F^\eta(r,s)-F(r,s)|\le\delta\|\vec{\mathbf{f}}'\|_\infty$ and the Lipschitz-continuity of F with respect to both of its variables), we get

$$|I_5+J_7|$$
$$=\Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}},k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,dp\,\rho_l(u_\epsilon(s,y)-k)\,dk\,dy\,ds$$
$$+E\int_Q\int_{\mathbb{R}}F^\eta(u_\epsilon,k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi\,dy\,ds\int_0^1\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,dk\,dp\Big|$$
$$=\Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}},u_\epsilon(s,y)-k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,dp\,\rho_l(k)\,dk\,dy\,ds$$
$$+E\int_Q\int_0^1\int_{\mathbb{R}}F^\eta(u_\epsilon,\hat{\mathbf{u}}(t,x,\alpha)-k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi\,dy\,ds\,\rho_l(k)\,dk\,dp\Big|$$
$$=\Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}},u_\epsilon(s,y)-k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,dp\,\rho_l(k)\,dk\,dy\,ds$$
$$-E\int_Q\int_0^1\int_{\mathbb{R}}F^\eta(u_\epsilon,\hat{\mathbf{u}}(t,x,\alpha)-k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,dy\,ds\,\rho_l(k)\,dk\,dp\Big|$$

$$= \left| E \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 [F^\eta(\hat{\mathbf{u}}, u_\epsilon(s,y) - k) - F^\eta(u_\epsilon, \hat{\mathbf{u}}(p) - k)] \nabla_x \rho_m(x-y) \right.$$
$$\left. \times \rho_n(t-s) \varphi \, dp \, \rho_l(k) \, dk \, dy \, ds \right|$$

and that $\lim_{\delta \to 0} \lim_{l \to \infty} \limsup_{n \to \infty} |I_5 + I_7| = 0$.

Let us now turn to the study of the integrals coming from the stochastic term. We start with the additional deterministic integrals coming from the Itô chain rule formula:

(7)
$$I_6 + J_8 = \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 h^2(\hat{\mathbf{u}}) \eta''(\hat{\mathbf{u}} - k) \rho_m(x-y) \rho_n(t-s)$$
$$\times \varphi \, dp \, \rho_l(u_\epsilon(s,y) - k) \, dk \, dy \, ds$$
$$+ \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_Q h^2(u_\epsilon) \eta''(u_\epsilon - k) \rho_m(x-y) \rho_n(t-s) \varphi \, dy \, ds$$
$$\times \int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha) - k) \, dk \, dp$$
$$\xrightarrow[n \to \infty]{} \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}) \eta''(\hat{\mathbf{u}} - k) \rho_m(x-y)$$
$$\times \varphi \, dp \, \rho_l(u_\epsilon(t,y) - k) \, dk \, dy$$
$$+ \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} h^2(u_\epsilon) \eta''(u_\epsilon - k) \rho_m(x-y) \varphi \, dy$$
$$\times \int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha) - k) \, dk \, dp$$
$$\xrightarrow[l \to \infty]{} \frac{1}{2} E \int_{\mathbb{R}^d} \int_Q \int_0^1 h^2(\hat{\mathbf{u}}) \eta''(\hat{\mathbf{u}} - u_\epsilon(t,y)) \rho_m(x-y) \varphi \, dp \, dy$$
$$+ \frac{1}{2} E \int_{\mathbb{R}^d} \int_Q \int_0^1 h^2(u_\epsilon) \eta''(u_\epsilon - \hat{\mathbf{u}}(t,x,\alpha)) \rho_m(x-y) \varphi \, dy \, dp,$$

where the passages to the limit follow from standard arguments (properties of convolution, Lebesgue points), using, moreover, the fact that η'' is bounded and Lipschitz-continuous. Note that it is not possible to pass to the limit with $\eta \to |\cdot|$ in the preceding term $\lim_l \lim_n I_6 + I_8$. Instead, we keep this term for the moment. We will combine it later on with corresponding integrals resulting from the stochastic integrals and show that the sum of these terms vanishes as $\eta = \eta_\delta \to |\cdot|$. For the convenience of the reader, let us just give the details of the arguments used to prove the above passages to the limit with n and then l to ∞.

- Limit as $n \to \infty$: set $\mathcal{G}_n^m = \rho_n(t-s)\rho_m(x-y)$ and $\mathcal{A}_k^l = \rho_l(\hat{\mathbf{u}}(p) - k)$; then, by a simple change of variables we have

$$\mathcal{A}_1 = \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) \eta''(\hat{\mathbf{u}}(p) - k) \mathcal{G}_n^m$$
$$\times \varphi(s,y) \, dp \, \rho_l(u_\epsilon(s,y) - k) \, dk \, dy \, ds$$
$$- \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}(p)) \eta''(\hat{\mathbf{u}}(p) - k) \rho_m(x-y)$$

$$\times \varphi(t,y)\rho_l(u_\epsilon(t,y)-k)\,dp\,dk\,dy$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\eta''(u_\epsilon(s,y)-k)\mathcal{G}_n^m\varphi(s,y)\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$-\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\rho_m(x-y)$$

$$\times \varphi(t,y)\,dy\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$=\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(s,y)+k)\mathcal{G}_n^m$$

$$\times \varphi(s,y)\,dp\,\rho_l(k)\,dk\,dy\,ds$$

$$-\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)$$

$$\times \rho_m(x-y)\varphi(t,y)\rho_l(k)\,dp\,dk\,dy$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q$$

$$\times [h^2(u_\epsilon(s,y))\eta''(u_\epsilon(s,y)-k)-h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)]$$

$$\times \mathcal{G}_n^m\varphi(s,y)\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)[\varphi(s,y)-\varphi(t,y)]\mathcal{G}_n^m\,dy\,ds$$

$$\times \int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\varphi(t,y)\left[1-\int_0^T\rho_n(t-s)\,ds\right]$$

$$\times \rho_m(x-y)\,dy\int_0^1\mathcal{A}_k^l\,dk\,dp,$$

$$\mathcal{A}_1=\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))[\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(s,y)+k)-\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)]$$

$$\times \mathcal{G}_n^m\varphi(s,y)\,dp\,\rho_l(k)\,dk\,dy\,ds$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)$$

$$\times \mathcal{G}_n^m[\varphi(s,y)-\varphi(t,y)]\,dp\,\rho_l(k)\,dk\,dy\,ds$$

$$-\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)\rho_m(x-y)\varphi(t,y)$$

$$\times \left[1-\int_0^T\rho_n(t-s)\,ds\right]\rho_l(k)\,dp\,dk\,dy$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\big[h^2(u_\epsilon(s,y))-h^2(u_\epsilon(t,y))\big]\eta''(u_\epsilon(s,y)-k)$$

$$\times\,\mathcal{G}_n^m\varphi(s,y)\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\big[\eta''(u_\epsilon(s,y)-k)-\eta''(u_\epsilon(t,y)-k)\big]h^2(u_\epsilon(t,y))$$

$$\times\,\mathcal{G}_n^m\varphi(s,y)\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)[\varphi(s,y)-\varphi(t,y)]$$

$$\times\,\mathcal{G}_n^m\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\varphi(t,y)$$

$$\times\left[1-\int_0^T\mathcal{G}_n^m\,ds\right]dy\int_0^1\mathcal{A}_k^l\,dk\,dp.$$

Therefore, as η'' is bounded and Lipschitz-continuous, we have

$$|\mathcal{A}_1|\le\frac{c}{2}E\int_Q\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\min(2\|\eta''\|_\infty,|u_\epsilon(s,y)-u_\epsilon(t,y)|)\mathcal{G}_n^m\varphi(s,y)\,dp\,dy\,ds$$

$$+\frac{c(\varphi)}{2n}E\int_Q\int_{\mathbb{R}}\int_K\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)\rho_m(x-y)\,dp\,\rho_l(k)\,dk\,dy$$

$$+\frac{1}{2}E\int_{T-2/n}^T\int_K\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)$$

$$\times\,\rho_m(x-y)\varphi(t,y)\rho_l(k)\,dp\,dk\,dy$$

$$+\frac{\|\eta''\|_\infty}{2}E\int_Q\int_{\mathbb{R}}\int_Q\big[h^2(u_\epsilon(s,y))-h^2(u_\epsilon(t,y))\big]\mathcal{G}_n^m\varphi(s,y)\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{c}{2}E\int_Q\int_{\mathbb{R}}\int_Q\min[2\|\eta''\|_\infty,|u_\epsilon(s,y)-u_\epsilon(t,y)|]h^2(u_\epsilon(t,y))$$

$$\times\,\mathcal{G}_n^m\varphi(s,y)\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{c(\varphi)}{2n}E\int_K\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\mathcal{G}_n^m\,dy\,ds\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$+\frac{1}{2}E\int_{\mathbb{R}^d}\int_{T-2/n}^T\int_{\mathbb{R}}\int_{\mathbb{R}^d}h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\varphi(t,y)$$

$$\times\,\rho_m(x-y)\,dy\int_0^1\mathcal{A}_k^l\,dk\,dp$$

$$
\begin{aligned}
&\leq \frac{c}{2} E \int_Q \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \mathcal{G}_n^m \varphi(s,y)\, dp\, dy\, ds \\
&\quad + \frac{c(\varphi,\eta'')}{2n} E \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p))\, dp + \frac{c(\varphi,\eta'')}{2} E \int_{T-2/n}^T \int_K \int_0^1 h^2(\hat{\mathbf{u}}(p))\, dp \\
&\quad + \frac{c(\varphi)\|\eta''\|_\infty}{2} E \int_Q \int_0^T \left[h^2(u_\epsilon(s,y)) - h^2(u_\epsilon(t,y))\right] \rho_n(t-s)\, dy\, ds\, dt \\
&\quad + \frac{c(\varphi)}{2} E \int_0^T \int_Q \min[2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|] h^2(u_\epsilon(t,y)) \rho_n(t-s)\, dy\, ds\, dt \\
&\quad + \frac{c(\varphi)\|\eta''\|_\infty}{2n} E \int_Q h^2(u_\epsilon(t,y))\, dy\, dt \\
&\quad + \frac{\|\eta''\|_\infty}{2} E \int_{T-2/n}^T \int_{\mathbb{R}^d} h^2(u_\epsilon(t,y)) \varphi(t,y)\, dy\, dt \\
&\leq \frac{c(m,\varphi)}{2} E \int_Q \int_K \int_0^T \int_0^1 h^2(\hat{\mathbf{u}}(p)) \\
&\quad \times \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \rho_n(t-s)\, dp\, dy\, ds \\
&\quad + \frac{c(\varphi)}{2} E \int_0^T \int_Q \min[2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|] h^2(u_\epsilon(t,y)) \rho_n(t-s)\, dy\, ds\, dt \\
&\quad + \frac{c(\varphi,\eta'')}{2} E \int_Q \int_0^T \left[h^2(u_\epsilon(s,y)) - h^2(u_\epsilon(t,y))\right] \rho_n(t-s)\, dy\, ds\, dt \\
&\quad + \frac{c(\varphi,\eta'')}{2} E \int_{T-2/n}^T \int_K \int_0^1 h^2(\hat{\mathbf{u}}(p)) + h^2(u_\epsilon(t,x))\, dp \\
&\quad + \frac{c(\varphi,\eta'')}{2n} E \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) + h^2(u_\epsilon(t,x))\, dp.
\end{aligned}
$$

Obviously, the last two integrals tend to 0 as $n \to \infty$. As to the first integral on the right, note that

$$
\begin{aligned}
&E \int_{\mathbb{R}^d} \int_K \int_0^1 \int_0^T \int_0^T h^2(\hat{\mathbf{u}}(p)) \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \rho_n(t-s)\, dp\, ds\, dy \\
&\quad = E \int_{\mathbb{R}^d} \int_K \int_0^1 \int_0^T h^2(\hat{\mathbf{u}}(p)) \\
&\quad \times \left[\int_0^T \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \rho_n(t-s)\, ds \right] dp\, dy \\
&\quad = E \int_{\mathbb{R}^d} \int_K \int_0^1 \int_0^T h^2(\hat{\mathbf{u}}(p)) A_n(t)\, dp\, dy,
\end{aligned}
$$

where $A_n(t) = \int_0^T \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|)\rho_n(t-s)\, ds$. For a.a. (ω, x, y, α) fixed, $s \mapsto u_\epsilon(s,\cdot)$ and $t \mapsto \hat{\mathbf{u}}(t,\cdot)$ are elements of $L^2(0,T)$. Therefore, $|A_n(t)| \leq \int_0^T |u_\epsilon(s,y) - u_\epsilon(t,y)|\rho_n(t-s)\, ds$ converges to 0 as $n \to \infty$ a.e. on $]0,T[$ (in every Lebesgue point), and thus $h^2(\hat{\mathbf{u}}(p))A_n(t)$ tends to 0 for a.a. $t \in]0,T[$. As

$|h^2(\hat{\mathbf{u}}(p))A_n(t)| \leq 2\|\eta''\|_\infty h^2(\hat{\mathbf{u}}(p))$, by Lebesgue's theorem, $\int_0^T h^2(\hat{\mathbf{u}}(p))A_n(t)\,dt$ tends to 0 as $n \to \infty$, for a.a. (ω, x, y, α). Then, using the same estimate, by Lebesgue's dominated convergence theorem we may conclude that the integral over $\Omega \times \mathbb{R}^d \times K \times (0,1)$ converges to 0. The second integral on the right can be shown to converge to 0 by using the same type of arguments. Finally, note that the third integral tends to 0 by the properties of convolution in $L^1(0,T;L^1(\Omega \times \mathbb{R}^d))$.

• Limit as $l \to \infty$: Again by a simple change of variables we have

$$\begin{aligned}
\mathcal{A}_2 := {} & \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y)\varphi(t,y)\,dp\,\rho_l(u_\epsilon(t,y)-k)\,dk\,dy \\
& + \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\rho_m(x-y)\varphi(t,y)\,dy \\
& \times \int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,dk\,dp \\
& - \frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y))\rho_m(x-y)\varphi(t,y)\,dp\,dy \\
& - \frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha))\rho_m(x-y)\varphi(t,y)\,dy\,dp \\
= {} & \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)\rho_m(x-y)\varphi(t,y)\,dp\,\rho_l(k)\,dk\,dy \\
& + \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha)+k) \\
& \times \rho_m(x-y)\varphi(t,y)\,dy\,\rho_l(k)\,dk\,dp \\
& - \frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y))\rho_m(x-y)\varphi(t,y)\,dp\,dy \\
& - \frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha))\rho_m(x-y)\varphi(t,y)\,dy\,dp \\
= {} & \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))[\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)-\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y))] \\
& \times \rho_m(x-y)\varphi(t,y)\,dp\,\rho_l(k)\,dk\,dy \\
& + \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 [\eta''(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha)+k)-\eta''(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha))] \\
& \times h^2(u_\epsilon(t,y))\rho_m(x-y)\varphi(t,y)\,dy\,\rho_l(k)\,dk\,dp.
\end{aligned}$$

Using once again the Lipschitz-continuity of η'' we obtain the estimate

$$\begin{aligned}
|\mathcal{A}_2| \leq {} & \frac{c(\eta'')}{2l}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\rho_m(x-y)\varphi(t,y)\,dp\,\rho_l(k)\,dk\,dy \\
& + \frac{c(\eta'')}{2l}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(u_\epsilon(t,y))\rho_m(x-y)\varphi(t,y)\,dy\,\rho_l(k)\,dk\,dp
\end{aligned}$$

$$\leq \frac{c(\eta'')}{2l} E \int_Q \int_{\mathbb{R}^d} \int_0^1 [h^2(\hat{\mathbf{u}}(p)) + h^2(u_\epsilon(t,y))]\rho_m(x-y)\varphi(t,y)\,dp\,dy$$

$$\xrightarrow[l\to\infty]{} 0.$$

Now we come to the estimate of the most interesting part, the stochastic integrals:

(8)
$$\begin{aligned} I_7 + J_9 &= E\int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\hat{\mathbf{u}} - k)h(\hat{\mathbf{u}})\,d\alpha\varphi\mathcal{G}_n^m\,dx\,dw(t)\rho_l(u_\epsilon(s,y)-k)\,dk\,dy\,ds \\ &\quad + E\int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\mathcal{G}_n^m\,dy\,dw(s) \int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,dk\,dp \\ &= E\int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\hat{\mathbf{u}} - k)h(\hat{\mathbf{u}})\,d\alpha\varphi\mathcal{G}_n^m\,dx\,dw(t)\rho_l(u_\epsilon(s,y)-k)\,dk\,dy\,ds, \end{aligned}$$

where $\mathcal{G}_n^m = \rho_m(x-y)\rho_n(t-s)$. Indeed, since $\mathrm{supp}\rho_n \subset \mathbb{R}^-$,

$$\begin{aligned} &E\int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\mathcal{G}_n^m\,dy\,dw(s) \int_0^1 \rho_l(\hat{\mathbf{u}}(p)-k)\,dk\,dp \\ &\quad = \int_Q \int_{\mathbb{R}} E\left[\int_t^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\mathcal{G}_n^m\,dy\,dw(s) \int_0^1 \rho_l(\hat{\mathbf{u}}(p)-k)\,d\alpha\right] dk\,dx\,dt = 0, \end{aligned}$$

since $\alpha(t) := \int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,d\alpha$ is predictable and if one denotes by $\beta(s) := \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\rho_m(x-y)\rho_n(t-s)\,dy$, one has that

$$\begin{aligned} E\left[\alpha(t)\int_t^T \beta(s)\,dw(s)\right] &= E\left[\alpha(t)\int_0^T \beta(s)\,dw(s)\right] - E\left[\alpha(t)\int_0^t \beta(s)\,dw(s)\right] \\ &= E\left[\alpha(t)\int_0^t \beta(s)\,dw(s)\right] - E\left[\alpha(t)\int_0^t \beta(s)\,dw(s)\right] = 0 \end{aligned}$$

and as

$$E\left[\alpha(t)\int_0^T \beta(s)\,dw(s)\right] = E\left[\alpha(t)E\left[\int_0^T \beta(s)\,dw(s)|F_t\right]\right] = E\left[\alpha(t)\int_0^t \beta(s)\,dw(s)\right].$$

By the same type of arguments for $\rho_l(u_\epsilon(s-2/n,y)-k)$ and $\int_{\mathbb{R}^d}\int_{s-2/n}^s\int_0^1 \eta'(\hat{\mathbf{u}} - k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dx\,dw(t)$ we deduce, since supp $\rho_n(\cdot - s) \subset [s-2/n, s]$, that

$$\begin{aligned} I_7 + J_9 &= E\int_Q \int_{\mathbb{R}} \int_{s-2/n}^s \int_{\mathbb{R}^d} \int_0^1 \eta'(\hat{\mathbf{u}} - k)h(\hat{\mathbf{u}})\,d\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,dx\,dw(t) \\ &\qquad \times \rho_l(u_\epsilon(s,y)-k)\,dk\,dy\,ds \\ &= E\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{s-2/n}^s \int_0^1 \eta'(\hat{\mathbf{u}} - k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx \\ &\qquad \times [\rho_l(u_\epsilon(s,y)-k) - \rho_l(u_\epsilon(s-2/n,y)-k)]\,dk\,dy\,ds \\ &= E\int_Q \int_{\mathbb{R}} G_{n,m}(\omega,s,y,k)[\rho_l(u_\epsilon(s,y)-k) - \rho_l(u_\epsilon(s-2/n,y)-k)]\,dk\,dy\,ds, \end{aligned}$$

where

$$y \mapsto G_{n,m}(\omega,s,y,k) = \int_{\mathbb{R}^d} \int_{s-2/n}^s \int_0^1 \eta'(\hat{\mathbf{u}} - k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx.$$

Remind at this point that u_ϵ is the solution of $u_\epsilon(0,\cdot) = u_0^\epsilon$ and

$$du_\epsilon = [\epsilon\Delta u_\epsilon + \operatorname{div} \vec{\mathbf{f}}(u_\epsilon)]\,dt + h(u_\epsilon)\,dw = A_\epsilon\,dt + h(u_\epsilon)\,dw.$$

Thanks to Appendix A.2, this solution is in $\mathcal{N}_w^2(0,T,H^2(\mathbb{R}^d))$ and

$$y \in \mathbb{R}^d \text{ a.e.}, \, du_\epsilon(\cdot,y) = A_\epsilon(\cdot,y)\,dt + h(u_\epsilon(\cdot,y))\,dw.$$

Therefore, by Itô's formula,

$$\begin{aligned}
&\rho_l(u_\epsilon(s,y)-k) - \rho_l(u_\epsilon(s-2/n,y)-k) \\
&\quad= \int_{s-\frac{2}{n}}^{s} \rho_l'(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma + \int_{s-\frac{2}{n}}^{s} \rho_l'(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\,dw(\sigma) \\
&\qquad+ \frac{1}{2}\int_{s-\frac{2}{n}}^{s} \rho_l''(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma \\
&\quad= -\frac{\partial}{\partial k}\left[\int_{s-\frac{2}{n}}^{s} \rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma + \int_{s-\frac{2}{n}}^{s} \rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\,dw(\sigma)\right. \\
&\qquad\left.+ \frac{1}{2}\int_{s-\frac{2}{n}}^{s} \rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
I_7 + J_9 &= E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx \\
&\qquad\times[\rho_l(u_\epsilon(s,y)-k) - \rho_l(u_\epsilon(s-2/n,y)-k)]\,dk\,dy\,ds \\
&= -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx \\
&\qquad\times\frac{\partial}{\partial k}\left[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma + \int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\,dw(\sigma)\right. \\
&\qquad\left.+\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma\right]dk\,dy\,ds \\
&= E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\partial}{\partial k}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx \\
&\qquad\times\left[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma + \int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\,dw(\sigma)\right. \\
&\qquad\left.+\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma\right]dk\,dy\,ds \\
&= -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx \\
&\qquad\times\left[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma + \int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\,dw(\sigma)\right. \\
&\qquad\left.+\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma\right]dk\,dy\,ds =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3.
\end{aligned}$$

We will prove that $\mathbb{I}_1$ and $\mathbb{I}_3$ tend to 0 as $n \to \infty$. To this end we estimate first $\mathbb{I}_1$. Using Cauchy–Schwarz inequality, Jensen inequality and the isometry property of the Itô integral, we find

$$
\begin{aligned}
|\mathbb{I}_1| = &\bigg| \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\left[\left[\int_{s-2/n}^{s} \int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\varphi\rho_m(x-y)\,dw(t)\right]\right. \\
&\times \left.\left[\int_{s-\frac{2}{n}}^{s} \rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma\right]\right]\,dx\,dk\,dy\,ds \bigg| \\
\le &\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left[E\left[\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\varphi\rho_m(x-y)\,dw(t)\right]^2\right]^{1/2} \\
&\times \left[E\left[\int_{s-\frac{2}{n}}^{s} \rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,d\sigma\right]^2\right]^{1/2} dx\,dk\,dy\,ds \\
\le &\frac{\sqrt{2}}{\sqrt{n}} \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left[E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,d\alpha\rho_n^2(t-s)\varphi^2\rho_m^2(x-y)\,dt\right]^{1/2} \\
&\times \left[E\int_{s-\frac{2}{n}}^{s} \rho_l^2(u_\epsilon(\sigma,y)-k)A_\epsilon^2(\sigma,y)\,d\sigma\right]^{1/2} dx\,dk\,dy\,ds \\
\le &C(\varphi)\frac{m^d nl}{\sqrt{n}} \int_{]0,T[}\int_K \int_{\|x-y\|<1/m}\int_{\mathbb{R}} \left[E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,d\alpha\,dt\right]^{1/2} \\
&\times \left[E\int_{s-\frac{2}{n}}^{s} 1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}A_\epsilon^2(\sigma,y)\,d\sigma\right]^{1/2} dk\,dx\,dy\,ds \\
\le &C(\varphi)m^d l\sqrt{n}\int_{]0,T[}\int_K \int_{\|x-y\|<1/m}\int_{\mathbb{R}} E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}(t,x)-k) \\
&\times h^2(\hat{\mathbf{u}}(t,x))\,dp\,dk\,dy\,ds \\
&+ C(\varphi)l\sqrt{n}\int_{]0,T[}\int_K\int_{\mathbb{R}} E\int_{s-\frac{2}{n}}^{s} 1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}A_\epsilon^2(\sigma,y)\,d\sigma\,dk\,dy\,ds \\
\le &C(\varphi,\eta)m^d l\sqrt{n}\int_{]0,T[}\int_K \int_{\|x-y\|<1/m} E\int_{s-2/n}^{s}\int_0^1 h^2(\hat{\mathbf{u}}(t,x))\,d\alpha\,dt\,dx\,dy\,ds \\
&+ C(\varphi)\sqrt{n}\int_{]0,T[}\int_K E\int_{s-\frac{2}{n}}^{s} A_\epsilon^2(\sigma,y)\,d\sigma\,dy\,ds \\
\le &\frac{C(\varphi,\eta)m^d l}{\sqrt{n}}\int_{]0,T[}\int_K \int_{\|x-y\|<1/m} E\int_0^1 h^2(\hat{\mathbf{u}}(s,x))\,d\alpha\,dx\,dy\,ds \\
&+ \frac{C(\varphi)}{\sqrt{n}}\int_{]0,T[}\int_K EA_\epsilon^2(s,y)\,dy\,ds \\
\le &\frac{C(\varphi,\eta)l}{\sqrt{n}}[\|h(\hat{\mathbf{u}})\|^2 + \|A_\epsilon\|^2] \to_{n\to+\infty} 0.
\end{aligned}
$$

Since, by Proposition A.5, $u_\epsilon \in L^4(Q\times\Omega)$ if $u_0^\epsilon \in L^4(\mathbb{R}^d)$, we can also prove that $\lim_{n\to+\infty} \mathbb{I}_3 = 0$. Indeed, using the same arguments as before, we have the following estimate on $\mathbb{I}_3$:

$$\begin{aligned}
|\mathbb{I}_3| &= \left| \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\left[\left[\int_{s-2/n}^s \int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\varphi\rho_m(x-y)\,dw(t)\right]\right.\right.\\
&\quad \left.\left.\times\left[\int_{s-\frac{2}{n}}^s \rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma\right]\right]dx\,dk\,dy\,ds\right|\\
&\le \int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left[E\left[\int_{s-2/n}^s\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,d\alpha\rho_n(t-s)\varphi\rho_m(x-y)\,dw(t)\right]^2\right]^{1/2}\\
&\quad \times\left[E\left[\int_{s-\frac{2}{n}}^s \rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,d\sigma\right]^2\right]^{1/2} dx\,dk\,dy\,ds\\
&\le \frac{\sqrt{2}}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left[E\int_{s-2/n}^s\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,d\alpha\rho_n^2(t-s)\varphi^2\rho_m^2(x-y)\,dt\right]^{1/2}\\
&\quad \times\left[E\int_{s-\frac{2}{n}}^s \rho_l'^2(u_\epsilon(\sigma,y)-k)h^4(u_\epsilon(\sigma,y))\,d\sigma\right]^{1/2} dx\,dk\,dy\,ds\\
&\le C(\varphi,l)\frac{m^d n}{\sqrt{n}}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}\int_{\mathbb{R}}\left[E\int_{s-2/n}^s\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,d\alpha\,dt\right]^{1/2}\\
&\quad \times\left[E\int_{s-\frac{2}{n}}^s 1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}h^4(u_\epsilon(\sigma,y))\,d\sigma\right]^{1/2} dk\,dx\,dy\,ds\\
&\le C(\varphi,l)m^d\sqrt{n}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}\int_{\mathbb{R}} E\int_{s-2/n}^s\int_0^1 \eta''^2(\hat{\mathbf{u}}(t,x)-k)\\
&\quad \times h^2(\hat{\mathbf{u}}(t,x))\,dp\,dk\,dy\,ds\\
&\quad + C(\varphi,l)\sqrt{n}\int_{]0,T[}\int_K\int_{\mathbb{R}} E\int_{s-\frac{2}{n}}^s 1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}h^4(u_\epsilon(\sigma,y))\,d\sigma\,dk\,dy\,ds\\
&\le C(\varphi,\eta,l)m^d\sqrt{n}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m} E\int_{s-2/n}^s\int_0^1 h^2(\hat{\mathbf{u}}(t,x))\,d\alpha\,dt\,dx\,dy\,ds\\
&\quad + C(\varphi,l)\sqrt{n}\int_{]0,T[}\int_K E\int_{s-\frac{2}{n}}^s h(u_\epsilon)^4(\sigma,y)\,d\sigma\,dy\,ds\\
&\le \frac{C(\varphi,\eta,l)m^d}{\sqrt{n}}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m} E\int_0^1 h^2(\hat{\mathbf{u}}(s,x))\,d\alpha\,dx\,dy\,ds\\
&\quad + \frac{C(\varphi,l)}{\sqrt{n}}\int_{]0,T[}\int_K Eh(u_\epsilon)^4(s,y)\,dy\,ds\\
&\le \frac{C(\varphi,\eta,l)}{\sqrt{n}}[\|h(\hat{\mathbf{u}})\|^2 + \|h(u_\epsilon)^2\|^2] \to_{n\to+\infty} 0.
\end{aligned}$$

It remains to consider $\mathbb{I}_2$. Using the properties of the stochastic Itô integral we find

$$\mathbb{I}_2 = -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)\,dw(t)\varphi\rho_m(x-y)\,dx$$
$$\times\left[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\,dw(\sigma)\right]dk\,dy\,ds$$
$$=-\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\left[\left[\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)\,dw(t)\right]\right.$$
$$\left.\times\left[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma)\right]\right]\varphi\rho_m(x-y)\,dx\,dk\,dy\,ds$$
$$=-\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\left[\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,dt\right]$$
$$\times\varphi\rho_m(x-y)\,dx\,dk\,dy\,ds$$

which is now a deterministic term, and then it follows again by standard arguments that

$$\mathbb{I}_2\underset{n}{\longrightarrow}-\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\Big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,dt\Big]$$
$$\times\varphi\rho_m(x-y)\,dx\,dk\,dy\,ds$$
$$\underset{l}{\longrightarrow}-\int_Q\int_{\mathbb{R}^d}E\Big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y))h(\hat{u}(t,x))h(u_\epsilon(t,y))\,dt\Big]\varphi\rho_m(x-y)\,dx\,dy\,ds.$$

Indeed, we have

- limit as $n\to\infty$:

$$\mathcal{A}_1:=-\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\left[\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,dt\right]$$
$$\times\varphi\rho_m(x-y)\,dx\,dk\,dy\,ds$$
$$+\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\left[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\right]$$
$$\times\varphi\rho_m(x-y)\,dx\,dk\,dy\,dt$$
$$=-\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$
$$\times\left[\int_0^T\varphi\rho_n(t-s)\,ds\right]\rho_m(x-y)\,dt\,dx\,dk\,dy$$
$$+\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$
$$\times\varphi\rho_m(x-y)\,dx\,dk\,dy\,dt$$

$$= \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$

$$\times \left[\varphi(t,y) - \int_0^T \varphi(s,y)\rho_n(t-s)\,ds\right]\rho_m(x-y)\,dt\,dx\,dk\,dy,$$

$$\mathcal{A}_1 = \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$

$$\times \int_0^T [\varphi(t,y)-\varphi(s,y)]\rho_n(t-s)\,ds\rho_m(x-y)\,dt\,dx\,dk\,dy$$

$$+ \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$

$$\times \varphi(t,y)\left[1 - \int_0^T \rho_n(t-s)\,ds\right]\rho_m(x-y)\,dt\,dx\,dk\,dy.$$

$$|\mathcal{A}_1| \le \frac{c(\varphi)}{n}\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$

$$\times \rho_m(x-y)\,dt\,dx\,dk\,dy$$

$$+ \int_{T-2/n}^T \int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\big]$$

$$\times \varphi(t,y)\rho_m(x-y)\,dt\,dx\,dk\,dy$$

$$\le \frac{c(\varphi)}{n}\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-u_\epsilon(t,y)+k)h(\hat{u})\rho_l(k)h(u_\epsilon(t,y))\big]$$

$$\times \rho_m(x-y)\,dt\,dx\,dk\,dy$$

$$+ \int_{T-2/n}^T \int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} E\big[\eta''(\hat{u}-u_\epsilon(t,y)+k)h(\hat{u})\rho_l(k)h(u_\epsilon(t,y))\big]$$

$$\times \varphi(t,y)\rho_m(x-y)\,dt\,dx\,dk\,dy$$

$$\le \frac{c(\varphi,\eta'')}{n}\int_Q\int_{\mathbb{R}^d} E\big[h(\hat{u})h(u_\epsilon(t,y))\big]\rho_m(x-y)\,dt\,dx\,dy$$

$$+ \int_{T-2/n}^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} E\big[h(\hat{u})h(u_\epsilon(t,y))\big]\varphi(t,y)\rho_m(x-y)\,dt\,dx\,dy \xrightarrow[n\to\infty]{} 0.$$

- limit as $l \to \infty$:

$$\mathcal{A}_2 = -\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,dt\big]$$

$$\times \varphi\rho_m(x-y)\,dx\,dk\,dy\,ds$$

$$+ \int_Q\int_{\mathbb{R}^d} E\big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y))h(\hat{u}(t,x))h(u_\epsilon(t,y))\,dt\big]\varphi\rho_m(x-y)\,dx\,dy\,ds$$

$$= -\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y)+k) - \eta''(\hat{u}(t,x)-u_\epsilon(t,y))\big]$$

$$\times h(\hat{u}(t,x))\rho_l(k)h(u_\epsilon(t,y))\varphi\rho_m(x-y)\,dx\,dk\,dy\,ds,$$

which can be estimated, using the Lipschitz-continuity of η'',

$$|\mathcal{A}_2| \le \frac{c(\eta'')}{l} \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} Eh(\hat{u}(t,x))\rho_l(k)h(u_\epsilon(t,y))\varphi\rho_m(x-y)\,dx\,dk\,dy\,ds$$

$$\le \frac{c(\eta'')}{l} \int_Q \int_{\mathbb{R}^d} Eh(\hat{u}(t,x))h(u_\epsilon(t,y))\varphi\rho_m(x-y)\,dx\,dy\,ds \xrightarrow[l\to\infty]{} 0.$$

Combining the preceding estimates yields that

$$\begin{aligned}
&\lim_l \lim_n [I_6 + J_8 + I_7 + J_9] \\
&= -\int_Q \int_{\mathbb{R}^d} E\big[\eta''(\hat{u}(t,x) - u_\epsilon(t,y))h(\hat{u}(t,x))h(u_\epsilon(t,y))\,dt\big]\varphi\rho_m(x-y)\,dx\,dy \\
&\quad + \frac{1}{2}E\int_Q \int_{\mathbb{R}^d} h^2(\hat{u})\eta''(\hat{u} - u_\epsilon(s,y))\rho_m(x-y)\varphi\,dx\,dy\,ds \\
&\quad + \frac{1}{2}E\int_Q \int_{\mathbb{R}^d} h^2(u_\epsilon)\eta''(u_\epsilon - \hat{u}(s,x))\rho_m(x-y)\varphi\,dy\,dx\,ds \\
&= \frac{1}{2}E\int_Q \int_{\mathbb{R}^d} [h(\hat{u}) - h(u_\epsilon)]^2\eta''(u_\epsilon - \hat{u}(s,x))\rho_m(x-y)\varphi\,dy\,dx\,ds \xrightarrow[\eta]{} 0
\end{aligned}$$

for $\eta = \eta_\delta \in \mathcal{E}$, the approximation of the absolute value function as defined above, since supp $\eta'' \subset [-\delta, \delta]$, and $|\eta''| \le \frac{2\pi}{\delta}$. Indeed, choosing this sequence of entropies yields the estimate

$$\begin{aligned}
&\left| E\int_Q \int_{\mathbb{R}^d} [h(\hat{u}) - h(u_\epsilon)]^2\eta''(u_\epsilon - \hat{u}(s,x))\rho_m(x-y)\varphi\,dy\,dx\,ds \right| \\
&\le c(h)E\int_Q \int_{\mathbb{R}^d} [\hat{u} - u_\epsilon]^2\eta''(u_\epsilon - \hat{u}(s,x))\rho_m(x-y)\varphi\,dy\,dx\,ds \\
&\le c(h)\delta^2 E\int_Q \int_{\mathbb{R}^d} \eta''(u_\epsilon - \hat{u}(s,x))\rho_m(x-y)\varphi\,dy\,dx\,ds \\
&\le c(h)\delta E\int_Q \int_{\mathbb{R}^d} \rho_m(x-y)\varphi\,dy\,dx\,ds \le c(h)\delta \int_Q \varphi\,dy\,ds \xrightarrow[\delta\to 0]{} 0.
\end{aligned}$$

Passing to the limits in $I_1 + \cdots + I_7 + J_1 + \cdots + J_9$ successively with n, l, η, ϵ and m, we thus obtain:

$$\begin{aligned}
0 \le E\int_{\mathbb{R}^d} |\hat{u}_0 - u_0|\,\varphi(0)\,dx + E\int_Q \int_0^1 \int_0^1 |\mathbf{u}(s,y,\beta) - \hat{\mathbf{u}}(s,y,\alpha)|\,\partial_s\varphi\,dy\,ds\,d\alpha\,d\beta \\
- E\int_Q \int_0^1 \int_0^1 F(\mathbf{u}(t,x,\beta), \hat{\mathbf{u}}(t,x,\alpha))\nabla\varphi\,dp\,d\beta. \qquad (4.1)
\end{aligned}$$

4.2. *Uniqueness and stability*

Proposition 4.4. *The measure-valued solution is unique. Moreover, it is the unique entropy solution.*

Proof. Note first that, thanks to a density argument, the Kato inequality still holds for any non-negative test-function $\varphi \in H^1(Q)$ with a compact support.

Set $\omega = \|f'\|_\infty$, $K > 0$ and denote by ψ any nonincreasing regular function with $1_{]-\infty,K]} \leq \psi \leq 1_{]-\infty,K+1]}$. Considering $\varphi(t,x) = \psi(|x| + \omega t)\gamma(t)$ in (4.1) leads to

$$\begin{aligned}
0 \leq \gamma(0)E &\int_{\mathbb{R}^d} |\hat{u}_0 - u_0|\, \psi(|x|)\, dx \\
&+ E \int_Q \gamma'(t) \int_0^1 \int_0^1 |\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}(t,x,\alpha)|\, \psi(|x| + \omega t)\, dp\, d\beta \\
&+ E \int_Q \int_0^1 \int_0^1 \omega\psi'(|x| + \omega t)\, |\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}(t,x,\alpha)|\, \gamma(t)\, dp\, d\beta \\
&- E \int_Q \int_0^1 \int_0^1 \psi'(|x| + \omega t) F(\mathbf{u}(t,x,\beta), \hat{\mathbf{u}}(t,x,\alpha)) \frac{x}{|x|} \gamma(t)\, dp\, d\beta.
\end{aligned}$$

As ψ is a nonincreasing function, the choice of ω yields

$$\begin{aligned}
-E &\int_Q \gamma'(t) \int_0^1 \int_0^1 |\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}(t,x,\alpha)|\, \psi(|x| + \omega t)\, dp\, d\beta \\
&\leq \gamma(0)E \int_{\mathbb{R}^d} |\hat{u}_0 - u_0|\, \psi(|x|)\, dx.
\end{aligned}$$

If $\hat{u}_0 = u_0$, $\gamma(t) = \frac{(T-t)^+}{T}$, for any positive R, fixing $K = R + \omega T$ leads to

$$E \int_{B(0,R)} \int_0^1 \int_0^1 |\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}(t,x,\alpha)|\; dp\, d\beta = 0.$$

This implies that, for any $R > 0$, $\mathbf{u}(t,x,\beta) = \hat{\mathbf{u}}(t,x,\alpha)$ for almost any $x \in B(0,R)$, $t \in]0,T[$, $\omega \in \Omega$, $\alpha, \beta \in]0,1[$. Thus, on the one hand $\mathbf{u} = \hat{\mathbf{u}}$; on the other hand $\mathbf{u}(t,x,\alpha) = u(t,x)$ is independent of α, hence an entropy solution in the sense of Definition 2.2. □

Proposition 4.5. *Entropy solutions satisfy a "contraction principle": if u_1, u_2 are entropy solutions of* (1.1) *corresponding to initial data $u_{1,0}, u_{2,0} \in L^2(\mathbb{R}^d)$, respectively, then, for any positive K and time t,*

$$E \int_{B(0,K-\omega t)} |u_1 - u_2|\, dx \leq \int_{B(0,K)} |u_{1,0} - u_{2,0}|\, dx.$$

Proof. This is a consequence of the previous proof when passing to the limit when ψ converges to $1_{]-\infty,K]}$. □

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Appendix A.

A.1. *Regularity of integrals with respect to parameters*

Denote by Λ a set of parameters, a bounded domain of $\mathbb{R}^d$ for example.

Theorem A.1 ([15, Theorem 7.6, p. 180]). *Suppose $p \geq 2$ and $rp > d$. Let $f_s(\lambda)$, $\lambda \in \Lambda$, be a predictable C_b^r-valued process satisfying, for any $|k| \leq r$, $\int_0^T \|D^k f_s\|_\infty^p dt < \infty$ a.s.*

Then the real valued stochastic integral $\int_0^t f_s(\lambda)dw(s)$ with parameter λ has a modification $L_t(\lambda)$ which satisfies the following properties:

$L_t(\lambda)$ is continuous in (t,λ) and l-times continuously differentiable in λ where $l < r - d/p$.

If $|k| < r - d/p$, then $D^k L_t(\Lambda)$ is continuous in (t,λ) and satisfies $D^k L_t(\Lambda) = \int_0^t D^k f_s(\lambda)dw(s)$ for any t and a.s.

Let us fix r and denote

$$f_n(y,s,k)(t) = \int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})d\alpha\varphi\rho_m(x-y)\rho_n(t-s)dx,$$

$$\begin{aligned} F_n(y,s,k) &= \int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})d\alpha\varphi\rho_m(x-y)\rho_n(t-s)dx\,dw(t) \\ &= \int_{s-2/n}^s\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})d\alpha\varphi\rho_m(x-y)\rho_n(t-s)dx\,dw(t). \end{aligned}$$

A classical result of differentiation yields $f_n \in C^\infty(Q\times\mathbb{R})$ and $f_n(y,s,k)$ is a predictable C_b^r-valued process for any r.

For a multi-index k, $|D^k f_n(s,y,k)(t)| \leq C(D^k\rho_m \otimes \rho_n \otimes \eta', \varphi)\int_0^1 |h(\hat{\mathbf{u}})|d\alpha$ and one gets that $E\int_0^T \|D^k f_n(s,y,k)(t)\|_\infty^2 dt < \infty$.

Therefore, F_n is a regular function and $D^k F_n = \int_0^T D^k f_n(\cdot)(\sigma)dw(\sigma)$.

A.2. *On the parabolic regularization*

The following result is a classic one. One can refer to [6, 11, 18] and many others authors. For the sequel of our purpose, we propose to refer to [24] but for the sake of convenience we propose to redevelop the proofs.

Proposition A.2. *For any positive ϵ, there exists a unique $u_\epsilon \in L^2(]0,T[\times\Omega, H^1(\mathbb{R}^d))$, $L^2(\mathbb{R}^d)$-adapted process to the filtration, with $\partial_t[u_\epsilon - \int_0^t h(u_\epsilon)dw] \in L^2(]0,T[\times\Omega, H^{-1}(\mathbb{R}^d))$ and such that u_ϵ is a weak solution of the stochastic nonlinear parabolic problem*

$$du_\epsilon - [\epsilon\Delta u_\epsilon - \operatorname{div}(\vec{\mathbf{f}}(u_\epsilon))]dt = h(u_\epsilon)dw \quad \text{in } \Omega\times\mathbb{R}^d\times]0,T[, \tag{A.1}$$

for the initial condition $u_0 \in L^2(\mathbb{R}^d)$. Moreover, there exists a positive constant C such that,

$$\forall\,\epsilon > 0, \quad \|u_\epsilon\|^2_{L^\infty[0,T;L^2(\Omega\times\mathbb{R}^d)]} + \epsilon\|u_\epsilon\|^2_{L^2[]0,T[\times\Omega;H_0^1(\mathbb{R}^d)]} \leq C.$$

Proof. Following [24], we propose a result of existence of a solution based on an implicit time discretization. The scheme is the following one:

For given small positive parameter Δt and u_n in $L^2(\Omega, H^1(\mathbb{R}^d))$, $\mathcal{F}_{n\Delta t}$-measurable, find u_{n+1} in $L^2(\Omega, H^1(\mathbb{R}^d))$, $\mathcal{F}_{(n+1)\Delta t}$-measurable, such that dP-a.s and for any v in $H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \left[(u_{n+1} - u_n)v + \Delta t\{\epsilon\nabla u_{n+1} \cdot \nabla v + \vec{\mathbf{f}}(u_{n+1}) \cdot \nabla v\}\right] dx$$
$$= (w_{n+1} - w_n) \int_{\mathbb{R}^d} h(u_n) v \, dx.$$

Lemma A.3. *If $\Delta t < \frac{2\epsilon}{\|\vec{\mathbf{f}}'\|_\infty^2}$, such a sequence (u_n) exists.*

Proof. Denote by $\mathbb{V} = L^2(\Omega, H^1(\mathbb{R}^d), \mathcal{F}_{(n+1)\Delta t}, dP)$, $\mathbb{L} = L^2(\Omega, L^2(\mathbb{R}^d), \mathcal{F}_{(n+1)\Delta t}, dP)$ and by T the application, defined for any $S \in \mathbb{H}$, by $u = T(S)$ is the solution in $\mathbb{V}$ of the variational problem: $\forall\, v \in \mathbb{V}$,

$$E\left[\int_{\mathbb{R}^d} \left[(u - u_n)v + \Delta t\{\epsilon\nabla u \cdot \nabla v + \vec{\mathbf{f}}(S) \cdot \nabla v\}\right] dx\right]$$
$$= E\left[(w_{n+1} - w_n) \int_{\mathbb{R}^d} h(u_n) v \, dx\right].$$

Thanks to the theorem of Lax–Milgram, T is a well-defined function. Moreover, for any $S_1, S_2 \in \mathbb{H}$, one has that

$$E\int_{\mathbb{R}^d} [|u_1 - u_2|^2 + \Delta t\epsilon|\nabla(u_1 - u_2)|^2] dx$$
$$= \Delta t E\int_{\mathbb{R}^d} (\vec{\mathbf{f}}(S_1) - \vec{\mathbf{f}}(S_2)) \cdot \nabla(u_1 - u_2) dx,$$

and

$$E\int_{\mathbb{R}^d} [|T(S_1) - T(S_2)|^2 + \Delta t\epsilon|\nabla(T(S_1) - T(S_2))|^2] dx$$
$$\leq \frac{\Delta t}{2\epsilon} E\int_{\mathbb{R}^d} (\vec{\mathbf{f}}(S_1) - \vec{\mathbf{f}}(S_2))^2 dx.$$

Thus, if $\Delta t < \frac{2\epsilon}{\|\vec{\mathbf{f}}'\|_\infty^2}$, T is a contractive mapping in $\mathbb{H}$ and the result holds.[a] □

Setting the test-function u_{n+1} yields

$$\frac{1}{2}E\int_{\mathbb{R}^d} [|u_{n+1}|^2 - |u_n|^2 + |u_{n+1} - u_n|^2] dx + \Delta t\epsilon E\int_{\mathbb{R}^d} |\nabla u_{n+1}|^2 dx$$
$$+ \Delta t E\int_{\mathbb{R}^d} \vec{\mathbf{f}}(u_{n+1}) \cdot \nabla u_{n+1} dx$$

[a]The variational equality holds a.s and for all $v \in H^1(\mathbb{R}^d)$ since $H^1(\mathbb{R}^d)$ is separable.

$$= E\left[(w_{n+1} - w_n)\int_{\mathbb{R}^d} h(u_n)[u_{n+1} - u_n]\,dx\right]$$
$$+ E\left[(w_{n+1} - w_n)\int_{\mathbb{R}^d} h(u_n)u_n\,dx\right]. \tag{A.2}$$

Note that $\int_{\mathbb{R}^d} \vec{\mathbf{f}}(u)\cdot\nabla u\,dx = 0$ for any $u \in D(\mathbb{R}^d)$, thus, for any $u \in H^1(\mathbb{R}^d)$. Then,

$$\frac{1}{2}E\int_{\mathbb{R}^d}[|u_{n+1}|^2 - |u_n|^2 + |u_{n+1} - u_n|^2]\,dx + \Delta t\epsilon E\int_{\mathbb{R}^d}|\nabla u_{n+1}|^2\,dx$$
$$\leq \Delta t E\int_{\mathbb{R}^d} h^2(u_n)\,dx + \frac{1}{4}E\int_{\mathbb{R}^d}[u_{n+1} - u_n]^2\,dx$$

and, if one denotes by $\|\cdot\|$ the norm in $L^2(\mathbb{R}^d)$,

$$\frac{1}{2}E\|u_n\|^2 + \frac{1}{4}\sum_{k=0}^{n-1}E\|u_{k+1} - u_k\|^2 + \Delta t\epsilon\sum_{k=0}^{n-1}E\|\nabla u_{k+1}\|^2$$
$$\leq \frac{1}{2}\|u_0\|^2 + \Delta t\sum_{k=0}^{n-1}E\|h(u_n)\|^2.$$

The discrete Gronwall lemma asserts then that

$$\frac{1}{2}E\|u_n\|^2 + \frac{1}{4}\sum_{k=0}^{n-1}E\|u_{k+1} - u_k\|^2 + \Delta t\epsilon\sum_{k=0}^{n-1}E\|\nabla u_{k+1}\|^2$$
$$\leq \frac{1}{2}\|u_0\|^2 + \|u_0\|^2\Delta t\|h'\|_\infty^2\sum_{k=0}^{n-1}e^{2\|h'\|_\infty^2 k\Delta t} \leq C.$$

This only gives an $L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))$ estimate on $u^{\Delta t}$ and an $L^2(\Omega\times Q)$ estimate on $\epsilon\nabla u^{\Delta t}$.

If $u_0 \in H^1(\mathbb{R}^d)$, setting the test-function $v = u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)$ yields:

$$\|u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)\|_{L^2(\mathbb{R}^d)}^2$$
$$+ \Delta t\epsilon\int_{\mathbb{R}^d}\nabla u_{n+1}\cdot\nabla[u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)]\,dx$$
$$= \Delta t\int_{\mathbb{R}^d}[u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)]\vec{\mathbf{f}}'(u_{n+1})\cdot\nabla u_{n+1}\,dx$$
$$\leq \frac{1}{2}\|u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2}C(\vec{\mathbf{f}}')(\Delta t)^2\|\nabla u_{n+1}\|_{L^2(\mathbb{R}^d)^d}^2.$$

Since $E(w_{n+1} - w_n)\int_{\mathbb{R}^d}\nabla u_n\cdot\nabla h(u_n)\,dx = 0$, one gets that

$$E\int_{\mathbb{R}^d}\nabla u_{n+1}\cdot\nabla[u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)]\,dx$$
$$= \frac{1}{2}E\Big[\|\nabla u_{n+1}\|_{L^2(\mathbb{R}^d)^d}^2 + \|\nabla(u_{n+1} - u_n)\|_{L^2(\mathbb{R}^d)^d}^2 - \|\nabla u_n\|_{L^2(\mathbb{R}^d)^d}^2\Big]$$

$$- E(w_{n+1} - w_n) \int_{\mathbb{R}^d} \nabla[u_{n+1} - u_n] \cdot \nabla h(u_n) dx$$

$$\geq \frac{1}{2} E \left[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d} + \frac{1}{2}\|\nabla(u_{n+1} - u_n)\|^2_{L^2(\mathbb{R}^d)^d} - \|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d} \right.$$

$$\left. - 2\Delta t \|\nabla h(u_n)\|^2_{L^2(\mathbb{R}^d)^d} \right].$$

And then,

$$E\|u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)\|^2_{L^2(\mathbb{R}^d)}$$

$$+ \Delta t \epsilon E \left[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d} - \|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d} + \frac{1}{2}\|\nabla(u_{n+1} - u_n)\|^2_{L^2(\mathbb{R}^d)^d} \right]$$

$$\leq 2(\Delta t)^2 \epsilon E\|\nabla h(u_n)\|^2_{L^2(\mathbb{R}^d)^d} + C(\vec{\mathbf{f}}')(\Delta t)^2 E\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}.$$

Consequently, for any k,

$$\sum_{n=0}^{k} \Delta t E \left\| \frac{u_{n+1} - u_n - (w_{n+1} - w_n)h(u_n)}{\Delta t} \right\|^2_{L^2(\mathbb{R}^d)}$$

$$+ \epsilon E\|\nabla u_{k+1}\|^2_{L^2(\mathbb{R}^d)^d} + \frac{\epsilon}{2} \sum_{n=0}^{k} E\|\nabla(u_{n+1} - u_n)\|^2_{L^2(\mathbb{R}^d)^d}$$

$$\leq C(\vec{\mathbf{f}}', h', \epsilon)\Delta t \sum_{n=0}^{k+1} E\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d} + \epsilon E\|\nabla u_0\|^2_{L^2(\mathbb{R}^d)^d} \leq Cte.$$

Denoting $\widetilde{u}^{\Delta t} = \sum_{k=1}^{N} [\frac{u_k - u_{k-1}}{\Delta t}[t - (k-1)\Delta t] + u_{k-1}] 1_{[(k-1)\Delta t, k\Delta t[}$ and $u^{\Delta t} = \sum_{k=1}^{N} u_k 1_{[(k-1)\Delta t, k\Delta t[}$, one gets that $u^{\Delta t}$ and $\widetilde{u}^{\Delta t}$ are bounded in $L^\infty(0, T, L^2(\Omega, H^1(\mathbb{R}^d)))$, $\partial_t[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}]$ is bounded in $L^2(0, T, L^2(\Omega, H^{-1}(\mathbb{R}^d)))$ and, if $u_0 \in H^1(\mathbb{R}^d)$, in $L^2(0, T, L^2(\Omega, L^2(\mathbb{R}^d)))$ where

$$\widetilde{B}^{\Delta t} = \sum_{k=1}^{N} \left[\frac{B_k - B_{k-1}}{\Delta t}[t - (k-1)\Delta t] + B_{k-1} \right] 1_{[(k-1)\Delta t, k\Delta t[}$$

with $B_n = \sum_{k=0}^{n-1} (w^{k+1} - w^k) h(u^k) = \int_0^{n\Delta t} h(u^{\Delta t}(\cdot - \Delta t)) dw$[b] and $\widetilde{u}^{\Delta t} - u^{\Delta t}$ converges to 0 in $L^2(0, T, L^2(\Omega, H^1(\mathbb{R}^d)))$.

Denote by u a limit point of $u^{\Delta t}$ and $\widetilde{u}^{\Delta t}$ for the weak-* convergence in $L^\infty(0, T, L^2(\Omega, H^1(\mathbb{R}^d)))$, h_u, respectively $\vec{\mathbf{f}}_u$, a limit point of $h(u^{\Delta t})$, respectively, $\vec{\mathbf{f}}(u^{\Delta t})$, for the weak convergence in $L^2(0, T, L^2(\Omega, H^1(\mathbb{R}^d)))$.

Since $\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}$ converges weakly in $L^2(\Omega, W(0, T))$ where $W(0, T)$ denotes the set of $L^2(0, T, H^1(\mathbb{R}^d))$-functions w such that $\partial_t w \in L^2(0, T, H^{-1}(\mathbb{R}^d))$ with the common identification of $L^2(\mathbb{R}^d)$ with its dual space, $\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}$ converges weakly

[b]We consider that $u^{\Delta t}(s) = u_0$ if $s < 0$.

in $L^2(\Omega, C([0,T], L^2(\mathbb{R}^d)))$. Thus, for any $t \in [0,T]$, $(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t})(t)$ converges weakly in $L^2(\Omega, L^2(\mathbb{R}^d))$.

Note that for $t \in [n\Delta t, (n+1)\Delta t[$, one has

$$\widetilde{B}^{\Delta t}(t) - \int_0^t h(u^{\Delta t}(s - \Delta t)\, dw(s) = (w^{n+1} - w^n)h(u^n)\frac{t - n\Delta t}{\Delta t} - (w(t) - w^n)h(u^n).$$

Then, thanks to the *a priori* estimates,

$$\begin{aligned} &E\left[\left\|(w^{n+1} - w^n)h(u^n)\frac{t - n\Delta t}{\Delta t} - (w(t) - w^n)h(u^n)\right\|^2\right] \\ &\quad = E[\|h(u^n)\|^2]\left[\frac{(t - n\Delta t)^2}{\Delta t} - 2\frac{t - n\Delta t}{\Delta t}(t - n\Delta t) + (t - n\Delta t)\right] \leq C\Delta t. \end{aligned}$$

Since $h(u^{\Delta t}(\cdot - \Delta t))$, as $h(u^{\Delta t})$, converges weakly to some function h_u in $L^2(0, T, L^2(\Omega, L^2(\mathbb{R}^d)))$, thanks to the properties of the Itô integral, $\int_0^{\cdot} h(u^{\Delta t}(s - \Delta t)\, dw(s)$ converges weakly to $\int_0^{\cdot} h_u\, dw(s)$ in $C([0,T], L^2(\Omega, L^2(\mathbb{R}^d)))$, and $\widetilde{B}^{\Delta t}$ does the same.

Thus, the weak convergence of $\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}$ is toward $u - \int_0^{\cdot} h_u\, dw(s)$ and, for any t, $\widetilde{u}^{\Delta t}(t)$ converges weakly in $L^2(\Omega, L^2(\mathbb{R}^d))$ to $u(t)$.

Moreover, for any $v \in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \partial_t[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}]v\, dx + \epsilon\int_{\mathbb{R}^d} \nabla u^{\Delta t}\nabla v\, dx + \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u^{\Delta t})\nabla v\, dx = 0$$

and at the limit, $u(0, \cdot) = u_0$ and,

$$\left\langle \partial_t\left[u - \int_0^t h_u\, dw(s)\right], v\right\rangle + \epsilon\int_{\mathbb{R}^d} \nabla u\nabla v\, dx + \int_{\mathbb{R}^d} \vec{\mathbf{f}}_u\nabla v\, dx = 0,$$

with the remark that one has an integral over $\mathbb{R}^d$ instead of the duality bracket if $u_0 \in H^1(\mathbb{R}^d)$.

Then, the Itô formula yields, for any positive c,

$$\begin{aligned} &e^{-ct}E\|u(t)\|^2 + 2\epsilon\int_0^t e^{-cs}E\|\nabla u\|^2\, ds + 2\int_0^t E\int_{\mathbb{R}^d} e^{-cs}\vec{\mathbf{f}}_u\nabla u\, dx\, ds \\ &\quad = \|u_0\|^2 - c\int_0^t e^{-cs}E\|u(s)\|^2\, ds + \int_0^t e^{-cs}E\|h_u\|^2\, ds. \end{aligned} \tag{A.3}$$

From (A.2), one has, for any positive c and $n > 0$, that

$$\begin{aligned} &E\int_{\mathbb{R}^d}[e^{-cn\Delta t}|u_{n+1}|^2 - e^{-c(n-1)\Delta t}|u_n|^2]\, dx + \Delta t 2\epsilon e^{-cn\Delta t}E\int_{\mathbb{R}^d}|\nabla u_{n+1}|^2\, dx \\ &\quad \leq \Delta t e^{-cn\Delta t}E\int_{\mathbb{R}^d} h^2(u_n)\, dx + [e^{-cn\Delta t} - e^{-c(n-1)\Delta t}]E\int_{\mathbb{R}^d}|u_n|^2\, dx. \end{aligned}$$

Adding from 0 to k, one gets that

$$\begin{aligned}&e^{-ck\Delta t}E\|u_{k+1}\|^2 + \Delta t 2\epsilon \sum_{n=0}^{k} e^{-cn\Delta t}E\|\nabla u_{n+1}\|^2 \\ &\quad \leq \|u_0\|^2 + \Delta t \sum_{k=0}^{k} e^{-cn\Delta t}E\|h(u_n)\|^2 - c\Delta t \sum_{n=1}^{k} e^{-c(n+1)\Delta t}E\|u_n\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&e^{-ck\Delta t}E\|u_{k+1}\|^2 + 2\epsilon \int_0^{(k+1)\Delta t} e^{-cs}E\|\nabla u^{\Delta t}\|^2 ds \\ &\quad \leq \|u_0\|^2 + \Delta t\|h(u_0)\|^2 + \int_0^{k\Delta t} e^{-cs}E\|h(u^{\Delta t})\|^2 ds \\ &\quad\quad - ce^{-c\Delta t}\int_0^{k\Delta t} e^{-cs}E\|u^{\Delta t}\|^2 ds.\end{aligned}$$

For $t \in [k\Delta t, (k+1)\Delta t[$, we obtain

$$\begin{aligned}&e^{-ct}E\|u^{\Delta t}(t)\|^2 + 2\epsilon \int_0^{t} e^{-cs}E\|\nabla u^{\Delta t}\|^2 ds \\ &\quad \leq \|u_0\|^2 + \Delta t\|h(u_0)\|^2 + \int_0^{t} e^{-cs}E\|h(u^{\Delta t})\|^2 ds \\ &\quad\quad - ce^{-c\Delta t}\int_0^{(t-\Delta t)^+} e^{-cs}E\|u^{\Delta t}\|^2 ds,\end{aligned}$$

and, since $u^{\Delta t}$ is bounded in $L^\infty(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$, one gets that

$$\begin{aligned}&e^{-ct}E\|u^{\Delta t}(t)\|^2 + 2\epsilon \int_0^{t} e^{-cs}E\|\nabla u^{\Delta t}\|^2 ds \\ &\quad \leq \|u_0\|^2 + \Delta t\|h(u_0)\|^2 + \int_0^{t} e^{-cs}E\|h(u^{\Delta t})\|^2 ds \\ &\quad\quad + ce^{-c\Delta t}\left[\int_{(t-\Delta t)^+}^{t} e^{-cs}E\|u^{\Delta t}\|^2 ds - \int_0^{t} e^{-cs}E\|u^{\Delta t}\|^2 ds\right] \\ &\quad \leq \|u_0\|^2 + C\Delta t + \int_0^{t} e^{-cs}E\|h(u^{\Delta t})\|^2 ds - ce^{-c\Delta t}\int_0^{t} e^{-cs}E\|u^{\Delta t}\|^2 ds.\end{aligned}$$

As, for any v in $H^1(\mathbb{R}^d)$ one has $\int_{\mathbb{R}^d} \vec{\mathbf{f}}(v)\nabla v\, dx = 0$,

$$\begin{aligned}&e^{-ct}E\|u^{\Delta t}(t)\|^2 + 2\epsilon \int_0^{t} e^{-cs}E\|\nabla(u^{\Delta t} - u)\|^2 ds \\ &\quad + 2\int_0^{t} e^{-cs}E\int_{\mathbb{R}^d}[\vec{\mathbf{f}}(u^{\Delta t}) - \vec{\mathbf{f}}(u)]\nabla[u^{\Delta t} - u]\, dx\, ds \\ &\quad + 4\epsilon\int_0^{t} e^{-cs}E\int_{\mathbb{R}^d}\nabla u^{\Delta t}\nabla u\, dx\, ds - 2\epsilon\int_0^{t} e^{-cs}E\|\nabla u\|^2 ds\end{aligned}$$

$$\leq \|u_0\|^2 + C\Delta t - 2\int_0^t e^{-cs} E \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u^{\Delta t})\nabla u \, dx\, ds$$
$$- 2\int_0^t e^{-cs} E \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u)\nabla u^{\Delta t} \, dx\, ds$$
$$+ 2\int_0^t e^{-cs} E \int_{\mathbb{R}^d} h(u^{\Delta t})h(u)\, dx\, ds - \int_0^t e^{-cs} E\|h(u)\|^2 ds$$
$$+ \int_0^t e^{-cs} E\|h(u^{\Delta t}) - h(u)\|^2 ds - ce^{-c\Delta t}\int_0^t e^{-cs} E\|u^{\Delta t} - u\|^2 ds$$
$$- 2ce^{-c\Delta t}\int_0^t e^{-cs} E\int_{\mathbb{R}^d} u^{\Delta t} u \, dx\, ds + ce^{-c\Delta t}\int_0^t e^{-cs} E\|u\|^2 ds.$$

Note that there exists $c = C(\vec{\mathbf{f}}, h, \epsilon) > 0$ such that, for Δt small, one has that

$$-2\epsilon\int_0^t e^{-cs} E\|\nabla(u^{\Delta t} - u)\|^2 ds - 2\int_0^t e^{-cs} E\int_{\mathbb{R}^d} [\vec{\mathbf{f}}(u^{\Delta t}) - \vec{\mathbf{f}}(u)]\nabla[u^{\Delta t} - u]\, dx\, ds$$
$$+ \int_0^t e^{-cs} E\|h(u^{\Delta t}) - h(u)\|^2 ds - ce^{-c\Delta t}\int_0^t e^{-s} E\|u^{\Delta t} - u\|^2 ds$$
$$\leq -\epsilon\int_0^t e^{-cs} E\|\nabla(u^{\Delta t} - u)\|^2 ds + \frac{1}{\epsilon}\int_0^t e^{-cs} E\|\vec{\mathbf{f}}(u^{\Delta t}) - \vec{\mathbf{f}}(u)\|^2 ds$$
$$+ \int_0^t e^{-cs} E\|h(u^{\Delta t}) - h(u)\|^2 ds - ce^{-c\Delta t}\int_0^t e^{-cs} E\|u^{\Delta t} - u\|^2 ds$$
$$\leq -\epsilon\int_0^t e^{-cs} E\|\nabla(u^{\Delta t} - u)\|^2 ds.$$

Thus,

$$\int_0^T e^{-ct} E\|u^{\Delta t}(t)\|^2 dt + \epsilon\int_0^T \int_0^t e^{-cs} E\|\nabla(u^{\Delta t} - u)\|^2 ds\, dt$$
$$\leq T\|u_0\|^2 + C\Delta t$$
$$- 2\int_0^T \int_0^t e^{-cs} E\int_{\mathbb{R}^d} \vec{\mathbf{f}}(u^{\Delta t})\nabla u\, dx\, ds\, dt - 2\int_0^T \int_0^t e^{-cs} E\int_{\mathbb{R}^d} \vec{\mathbf{f}}(u)\nabla u^{\Delta t}\, dx\, ds\, dt$$
$$+ 2\int_0^T \int_0^t e^{-cs} E\int_{\mathbb{R}^d} h(u^{\Delta t})h(u)\, dx\, ds\, dt - \int_0^T \int_0^t e^{-cs} E\|h(u)\|^2 ds\, dt$$
$$- 2ce^{-c\Delta t}\int_0^T \int_0^t e^{-cs} E\int_{\mathbb{R}^d} u^{\Delta t} u\, dx\, ds\, dt + ce^{-c\Delta t}\int_0^T \int_0^t e^{-cs} E\|u\|^2 ds\, dt$$
$$- 4\epsilon\int_0^T \int_0^t e^{-cs} E\int_{\mathbb{R}^d} \nabla u^{\Delta t}\nabla u\, dx\, ds\, dt + 2\epsilon\int_0^T \int_0^t e^{-cs} E\|\nabla u\|^2 ds\, dt. \tag{A.4}$$

This yields

$$\begin{aligned}
&\limsup_{\Delta t}\int_0^T e^{-ct}E\|u^{\Delta t}(t)\|^2\,dt\\
&\quad\leq \int_0^T\left[\|u_0\|^2-2\int_0^t e^{-cs}E\int_{\mathbb{R}^d}\vec{\mathbf{f}}_u\nabla u\,dx\,ds-2\epsilon\int_0^t e^{-cs}E\|\nabla u\|^2\,ds\right.\\
&\qquad\left.-c\int_0^t e^{-cs}E\|u\|^2\,ds\right]dt\\
&\qquad+2\int_0^T\int_0^t e^{-cs}E\int_{\mathbb{R}^d}h_u h(u)\,dx\,ds\,dt-\int_0^T\int_0^t e^{-cs}E\|h(u)\|^2\,ds\,dt,
\end{aligned}$$

and, thanks to (A.3),

$$\begin{aligned}
&\limsup_{\Delta t}\int_0^T e^{-ct}E\|u^{\Delta t}(t)\|^2\,dt+\int_0^T\int_0^t e^{-cs}E\|h_u-h(u)\|^2\,ds\,dt\\
&\quad\leq\int_0^T e^{-ct}E\|u(t)\|^2\,dt.
\end{aligned}$$

Thus, one gets that $h_u=h(u)$, $u^{\Delta t}$ converges to u in $L^2(]0,T[\times\Omega\times\mathbb{R}^d)$ and $\vec{\mathbf{f}}_u=\vec{\mathbf{f}}(u)$. This means that u is a solution.

Remark that it is a direct proof to show that it is unique.

Then, the stochastic energy asserts that (see for example Grecksch [11, Theorem 3.4, p. 42]):

$$\begin{aligned}
&\|u(t)\|^2_{L^2(\mathbb{R}^d)}+2\int_0^t\int_{\mathbb{R}^d}[\epsilon|\nabla u|^2+\vec{\mathbf{f}}(u)\cdot\nabla u]\,dx\,ds\\
&\quad=\|u(0)\|^2_{L^2(\mathbb{R}^d)}+2\int_0^t\int_{\mathbb{R}^d}uh(u)\,dx\,dw_s+\int_0^t\int_{\mathbb{R}^d}h^2(u)\,dx\,ds.
\end{aligned}$$

Since $\int_0^t\int_{\mathbb{R}^d}\vec{\mathbf{f}}(u_n)\cdot\nabla u_n\,dx\,ds=0$, taking the expectation and using the lemma of Gronwall yield the result of Proposition A.2. □

Corollary A.4. *If, moreover,* $u_0\in H^1(\mathbb{R}^d)$, *then*

$$\partial_t\left[u-\int_0^t h(u)\,dw\right]\in L^2(]0,T[\times\Omega,L^2(\mathbb{R}^d))\quad\textit{and}\quad\Delta u\in L^2(]0,T[\times\Omega,L^2(\mathbb{R}^d)).$$

Proof. The first part of the corollary is in the previous proof.

Then, one gets that $-\epsilon\Delta u=\operatorname{div}\vec{\mathbf{f}}(u)-\partial_t(u-\int_0^t h(u)\,dw)\in L^2(]0,T[\times\Omega\times\mathbb{R}^d)$. □

Proposition A.5. *If the initial condition* $u_0\in L^{2p}(\mathbb{R}^d)$, $p\geq 1$, *then* $u\in L^\infty(0,T,L^{2p}(\Omega\times\mathbb{R}^d))$ *as well.*

Proof. For any positive k, denote by ϕ_k the even function such that

$$\phi_k(x) = \begin{cases} x^{2p}, & \text{if } 0 \leq x < k, \\ p(2p-1)k^{2(p-1)}x^2 - 4p(p-1)k^{2p-1}x + (p-1)(2p-1)k^{2p}, & \text{if } k \leq x. \end{cases}$$

ϕ_k is a C^2-convex function and ϕ_k' is a Lipschitz-continuous function with $\phi_k'(0) = 0$. Thus, for any positive x, one gets $0 \leq \phi_k'(x) \leq c(k)x$ and $0 \leq \phi_k(x) = \int_0^x \phi_k'(s)\,ds \leq \frac{c(k)}{2}x^2$. This yields $E\int_{\mathbb{R}^d} \phi_k(u(t))\,dx < \infty$.

Lemma A.6. *For any $x \in \mathbb{R}$, on has $0 \leq x^2\phi_k''(x) \leq 2(2p-1)^2\phi_k(x)$.*

Proof. If $|x| < k$, $x^2\phi_k''(x) = 2p(2p-1)x^{2p} \leq 2(2p-1)^2\phi_k(x)$ since $p \leq 2p-1$.
If $|x| \geq k$,

$$\begin{aligned} &x^2\phi_k''(x) - 2(2p-1)^2\phi_k(x) \\ &\quad= 2p(2p-1)x^2k^{2(p-1)} - 2(2p-1)^2[p(2p-1)k^{2(p-1)}x^2 - 4p(p-1)k^{2p-1}x \\ &\qquad+ (p-1)(2p-1)k^{2p}] \\ &\quad= 8p^2(2p-1)(1-p)k^{2(p-1)}x^2 + 8(2p-1)^2p(p-1)k^{2p-1}x \\ &\qquad- 2(2p-1)^3(p-1)k^{2p} \\ &\quad= 2(2p-1)(p-1)k^{2(p-1)}[-4p^2x^2 + 4(2p-1)pkx - (2p-1)^2k^2] \leq 0. \end{aligned}$$

□

Thanks to the Itô formula, one gets that

$$\begin{aligned} &E\int_{\mathbb{R}^d} \phi_k(u(t))\,dx + \epsilon E\int_0^t\int_{\mathbb{R}^d} \phi_k''(u)|\nabla u|^2\,dx\,ds + E\int_0^t\int_{\mathbb{R}^d} \vec{\mathbf{f}}(u).\nabla\phi_k'(u)\,dx\,ds \\ &\quad= E\int_{\mathbb{R}^d} \phi_k(u_0)\,dx + \frac{1}{2}E\int_0^t\int_{\mathbb{R}^d} h^2(u)\phi_k''(u)\,dx\,ds. \end{aligned}$$

Since $\phi_k'' \geq 0$ and $E\int_0^t\int_{\mathbb{R}^d} \vec{\mathbf{f}}(u).\nabla\phi_k'(u)\,dx\,ds = 0$, one has that

$$E\int_{\mathbb{R}^d} \phi_k(u(t))\,dx \leq C + \frac{1}{2}E\int_0^t\int_{\mathbb{R}^d} h^2(u)\phi_k''(u)\,dx\,ds.$$

Then, assumptions on h and the previous lemma yield

$$E\int_{\mathbb{R}^d} \phi_k(u(t))\,dx \leq C + C(h,p)E\int_0^t\int_{\mathbb{R}^d} \phi_k(u)\,dx\,ds.$$

Thanks to Gronwall's lemma, $E\int_{\mathbb{R}^d} \phi_k(u(t))\,dx$ is bounded, independently of k and at the limit when k goes to infinity the theorem of Beppo Levi yields the proof.

□

A.3. *A basic reminder of Young measures*

A.3.1. *In finite-measure spaces*

In this section, we recall some basic facts on Young measures and refer to Balder [1], Castaing *et al.* [3], Saadoune and Valadier [19] and Valadier [22] for an abstract setting on the convergence of Young measures; and to DiPerna [8], Eymard *et al.* [9], Panov [17], Szepessy [20] and Tartar [21] for an application to nonlinear PDE.

Consider the space $L^1(\Theta,\mu,\mathbb{R})$ where $(\Theta,\mathcal{F},\mu)$ is a measure space with a positive bounded measure μ.

For u in $L^1(\Theta,\mu,\mathbb{R})$, the Young measure associated with u is τ_u, the measure on $\Theta\times\mathbb{R}$ image of μ by $x\mapsto(x,u(x))$.

A general Young measure τ is a positive measure on $\Theta\times\mathbb{R}$ such that, for any A in $\mathcal{F}$, $\tau(A\times\mathbb{R})=\mu(A)$.

A Young measure τ is described by its disintegration which is the unique family of probability measures on $\mathbb{R}$, $(d\tau_x)_{x\in\Theta}$, such that for any τ-measurable function ψ,

$$x\mapsto\int_{\mathbb{R}}\psi(x,\lambda)\,d\tau_x(\lambda)\text{ is }\mu\text{-measurable on }\Theta\text{ and}$$

$$\text{if }\psi\geq 0,\ \int_{\Theta\times\mathbb{R}}\psi\,d\tau=\int_\Theta\int_{\mathbb{R}}\psi(x,\lambda)\,d\tau_x(\lambda)\mu(dx).$$

Therefore, if $\tau=\tau_u$ is the Young measure associated with the above function u, then $\tau_x=\delta_{u(x)}$, the Dirac mass at $u(x)$.

Another way to define Young measures on $\Theta\times\mathbb{R}$ is to consider $\widetilde{\mathbf{u}}$ the notion of entropy process proposed by Gallouët *et al.* [9] or $\mathbf{u}$ the strong measure-valued solution proposed by Panov [17]. For a Young measure τ on $\Theta\times\mathbb{R}$ and F_x the left-continuous repartition function of τ_x, the functions $\widetilde{\mathbf{u}}$ and $\mathbf{u}$ are defined in $\Theta\times]0,1[$ by:

$$\widetilde{\mathbf{u}}(x,\alpha)=\sup\{t\in\mathbb{R},F_x(t)<\alpha\},\quad \mathbf{u}(x,\alpha)=\inf\{t\in\mathbb{R},F_x(t)>\alpha\}. \tag{A.5}$$

For fixed x, the two functions differ only on a countable set. Each author proves that the function is a $\mu\times\mathcal{L}$ measurable function on $\Theta\times]0,1[$ and for any positive Carathéodory function ψ,

$$\begin{aligned}\int_\Theta\int_{\mathbb{R}}\psi(x,\lambda)\,d\tau_x(\lambda)\mu(dx)&=\int_\Theta\int_0^1\psi(x,\mathbf{u}(x,\alpha))d\alpha\ \mu(dx)\\&=\int_\Theta\int_0^1\psi(x,\widetilde{\mathbf{u}}(x,\alpha))d\alpha\ \mu(dx).\end{aligned}$$

A sequence of Young measures $(\tau^n)_n$ is said to converge narrowly towards τ if $\int_{\Theta\times\mathbb{R}}\psi\,d\tau^n$ converges towards $\int_{\Theta\times\mathbb{R}}\psi\,d\tau$ for all bounded Carathéodory function ψ.

Consider now $(u_n)_n\subset L^1(\Theta,\mu,\mathbb{R})$ and denote by τ^n the associated Young measures.

If the sequence $(u_n)_n$ is assumed to be bounded in $L^1(\Theta)$, the theorem of Prohorov for Young measures (Balder [1], Saadoune *et al.* [19] and Valadier [22]) ensures that a subsequence $(\tau^{n_k})_k$ of $(\tau^n)_n$ and a Young measure τ exists such that τ^{n_k} converges narrowly towards τ.

Moreover:

(i) for μ-a.e. x in Θ, $\text{supp}(d\tau_x) \subset \bigcap_{p=1}^{\infty} \overline{\bigcup_{n\geq p}\{u_n(x)\}}$,

(ii) for any Carathéodory function ψ such that the sequence of functions $\{\psi(\cdot, u_n(\cdot))\}_n$ is uniformly integrable,

$$\int_\Theta \psi(x, u_n(x))\mu(dx) \to \int_{\Theta\times\mathbb{R}} \psi(x,\lambda)\,d\tau$$

(if the sequence $(u_n)_n$ is uniformly integrable, the above convergence still holds if one assumes that $|\psi(x,\lambda)| \leq \alpha(x) + k|\lambda|$ where $k \geq 0$ and $\alpha \in L^1(\Theta)$),

(iii) for any measurable function ψ, l.s.c. with respect to its second variable and such that $\{\psi(\cdot, u_n(\cdot))^-\}_n$ is uniformly integrable,

$$\liminf_{n\to\infty} \int_\Theta \psi(x, u_n(x))\mu(dx) \geq \int_{\Theta\times\mathbb{R}} \psi(x,\lambda)\,d\tau.$$

As a consequence, if u_n converges weakly to some u in L^1, it converges strongly to u in L^1, if and only if τ^n converges narrowly to τ_u (i.e. if $\mathbf{u}$, or respectively, $\widetilde{\mathbf{u}}$, is independent of α).

A.3.2. *In* $Q \times \Omega$

Consider in the sequel a bounded sequence (u_n) in $\mathcal{N}_w^2(0,T;L^2(\mathbb{R}^d))$.

For any $M > 0$, if one denotes by $Q_M =]0,T[\times B(0,M)$, the sequence is bounded in $L^2(Q_M \times \Omega)$ and it converges, up to a subsequence still denoted (u_n), in the sense of the Young measures in $Q_M \times \Omega$ to a given τ^M.

Setting $K > M$, a new subsequence converges in the sense of the Young measures in $Q_K \times \Omega$ to τ^K. Thus, for any $v \in L^1(Q_M \times \Omega)$ (extended by 0 in $[Q_K \backslash Q_M] \times \Omega$) and any bounded continuous function f, one gets that

$$\int_{Q_M\times\Omega} v\left[\int_{\mathbb{R}} f(\lambda)\,d\tau^M(\lambda) - \int_{\mathbb{R}} f(\lambda)\,d\tau^K(\lambda)\right] dx\,dt\,dP = 0.$$

Thus, for any bounded continuous function f, $\int_{\mathbb{R}} f(\lambda)\,d\tau^M(\lambda) = \int_{\mathbb{R}} f(\lambda)\,d\tau^K(\lambda)$ in $Q_M \times \Omega$ and $\tau^M = \tau^K$ restricted to $Q_M \times \Omega$.

Therefore, by a diagonal extraction of subsequences, there exists a Young measure τ on $Q \times \Omega \times \mathbb{R}$ such that, if $\psi : (t,x,\omega,\lambda) \in Q \times \Omega \times \mathbb{R} \mapsto \psi(t,x,\omega,\lambda)$ such that $\psi(.,u_n)$ is uniformly integrable, then

$$E\int_Q \psi(\cdot, u_n)\,dt\,dx \to E\int_Q\int_{\mathbb{R}} \psi(\cdot,\lambda)\,d\tau_{(t,x,\omega)}(\lambda)\,dt\,dx.$$

To prove it, recall that a bounded sequence $\psi(\cdot,u_n)$ in $L^1(Q\times\Omega)$ is uniformly integrable when (denote $\mathcal{L}^m$ the measure of Lebesgue in $\mathbb{R}^m$)

(1) $\forall\,\epsilon>0,\ \exists\,\delta>0,\ \mathcal{L}^{d+1}\otimes P(A)<\delta \Rightarrow \sup_n\int_A|\psi(\cdot,u_n)|dx\,dt\,dP<\epsilon$.
(2) $\forall\,\epsilon>0,\ \exists\,M_\epsilon>0,\ \sup_n\int_{[Q\setminus Q_{M_\epsilon}]\times\Omega}|\psi(\cdot,u_n)|dx\,dt\,dP<\epsilon$.[c]

Set $\epsilon>0$. Thanks to the first item of the uniform integrability, for any positive K, $|\psi(\cdot,u_n)|$ is uniformly integrable in $Q_K\times\Omega$ and

$$\int_{Q_K\times\Omega}|\psi(\cdot,u_n)|dx\,dt\,dP\to\int_{Q_K\times\Omega\times\mathbb{R}}|\psi|d\tau.$$

In particular, for any $K>M_\epsilon$,

$$\int_{[Q_K\setminus Q_{M_\epsilon}]\times\Omega}|\psi(\cdot,u_n)|dx\,dt\,dP\to\int_{[Q_K\setminus Q_{M_\epsilon}]\times\Omega\times\mathbb{R}}|\psi|d\tau$$

and, for any $K>M_\epsilon$,

$$\int_{[Q_K\setminus Q_{M_\epsilon}]\times\Omega\times\mathbb{R}}|\psi|d\tau\le\epsilon.$$

The theorem of Beppo Levy yields

$$\int_{[Q\setminus Q_{M_\epsilon}]\times\Omega\times\mathbb{R}}|\psi|d\tau\le\epsilon.$$

Using the notation $\mathbf{u}$, one gets that $\psi(\cdot,\mathbf{u})\in L^1(Q\times\Omega\times]0,1[)$ and

$$\begin{aligned}&\left|\int_{Q\times\Omega}\psi(\cdot,u_n)dx\,dt\,dP-\int_{Q\times\Omega\times]0,1[}\psi(\cdot,\mathbf{u})d\alpha\,dx\,dt\,dP\right|\\&\quad\le\left|\int_{Q_{M_\epsilon}\times\Omega}\psi(\cdot,u_n)dx\,dt\,dP-\int_{Q_{M_\epsilon}\times\Omega\times]0,1[}|\psi(\cdot,\mathbf{u})|d\alpha\,dx\,dt\,dP\right|\\&\qquad+\int_{[Q\setminus Q_{M_\epsilon}]\times\Omega}|\psi(\cdot,u_n)|dx\,dt\,dP+\int_{[Q\setminus Q_{M_\epsilon}]\times\Omega\times]0,1[}|\psi(\cdot,\mathbf{u})|d\alpha\,dx\,dt\,dP.\end{aligned}$$

Then,

$$\limsup_n\left|\int_{Q\times\Omega}\psi(\cdot,u_n)dx\,dt\,dP-\int_{Q\times\Omega\times]0,1[}\psi(\cdot,\mathbf{u})d\alpha\,dx\,dt\,dP\right|\le2\epsilon$$

and the result holds since the above inequality is satisfied for any $\epsilon>0$.

Assume that $(\psi(t,x,\omega,\lambda))$ is bounded in $L^p(Q\times\Omega)$ for a given $p\in]1,+\infty]$. Then, one gets that $\psi(\cdot,u_n)$ converges weakly (respectively, *-weakly if $p=+\infty$) to $\int_0^1\psi(\cdot,\mathbf{u})d\alpha$ in $L^p(Q\times\Omega)$.

Indeed, up to a subsequence $\psi(\cdot,u_{n_k})$ converges weakly in $L^p(Q\times\Omega)$ (respectively, *-weakly if $p=+\infty$) to an element called χ.

[c] This is needed since $\mathcal{L}^d$ is not finite on $\mathbb{R}^d$. This condition is useless when one considers the uniform integrability in a bounded measure space.

But, for any $\varphi \in L^q(Q\times\Omega)$ where q is the conjugate of p, $(\varphi\psi(\cdot,u_n))$ is uniformly integrable.[d]

Thus, at the limit, $\int_{Q\times\Omega}\varphi\chi\,dt\,dx\,dP = \int_{Q\times\Omega\times]0,1[}\psi(\cdot,\mathbf{u})\,d\alpha\varphi\,dt\,dx\,dP$. Then the limit is identified and the subsequence is not needed anymore.

In particular, if (u_n) is a bounded sequence in $L^p(Q\times\Omega)$ for a given $p\in]1,+\infty]$, then, $\mathbf{u}\in L^p(Q\times\Omega\times]0,1[)$.

A.3.3. *Predictability and Itô integral*

Let us revisit the measurability of $\mathbf{u}$ with respect to all variables (t,x,ω,α) as proposed by Panov.

Since for any $f\in C_b(\mathbb{R})$, $f(u_n)$ converges to $\int_{\mathbb{R}} f(\lambda)\,d\nu_{(t,x,\omega)}$ in $L^\infty(Q\times\Omega)$ weak-*, one gets that $\int_{\mathbb{R}} f(\lambda)\,d\nu_{(t,x,\omega)}$ is a $\mathbb{R}$-valued $\mathcal{L}^{d+1}\otimes P$ measurable function.

Therefore, $\int_{\mathbb{R}} f(\lambda)\,d\nu_{(t,x,\omega)}$ is a $\mathbb{R}$-valued $\mathcal{L}^{d+1}\otimes P$ measurable function for any bounded f in the union of Baire classes of continuous functions, thus for any Borel function. In particular, for any $c\in\mathbb{R}$, $(t,x,\omega)\mapsto\nu_{(t,x,\omega)}(]-\infty,c[)$ is measurable.

Let us recall that $\mathbf{u}(t,x,\omega,\alpha)=\inf\{c,\ \nu_{(t,x,\omega)}(]-\infty,c[)>\alpha\}$.

Set $\mu\in\mathbb{R}$ and denote by

$$E_\mu=\{(t,x,\omega,\lambda),\ \mathbf{u}(t,x,\omega,\lambda)<\mu\}\ \text{ and }\ F_\mu=\{(t,x,\omega,\lambda),\ \nu_{(t,x,\omega)}(]-\infty,\mu[)>\lambda\}.$$

Consider $(t,x,\omega,\lambda)\in E_\mu$. Then, $\mathbf{u}(t,x,\omega,\lambda)<\mu$ and by definition of the infimum, there exists $c\in]\mathbf{u}(t,x,\omega,\lambda),\mu[$ such that $\nu_{(t,x,\omega)}(]-\infty,c[)>\lambda$ and thus $\nu_{(t,x,\omega)}(]-\infty,\mu[)>\lambda$.

Consider $(t,x,\omega,\lambda)\in F_\mu$. Then, $\nu_{(t,x,\omega)}(]-\infty,\mu[)>\lambda$ and by left-continuity of the repartition function, there exists $c<\mu$ with $\nu_{(t,x,\omega)}(]-\infty,c[)>\lambda$. Then, by definition of $\mathbf{u}$, $\mathbf{u}(t,x,\omega,\lambda)\le c<\mu$.

Thus, $E_\mu=F_\mu$ and $\mathbf{u}$ is measurable since F_μ is measurable for any μ. Note that $(t,x,\omega)\mapsto\nu_{(t,x,\omega)}(]-\infty,\mu[)$ is also measurable for the σ-field $\mathcal{P}_T\times L(\mathbb{R}^d)$ where we recall that $\mathcal{P}_T$ denotes the predictable σ-field and $L(\mathbb{R}^d)$ the Lebesgue's one. So, $\mathbf{u}$ is measurable for the σ-field $\mathcal{P}_T\times L(\mathbb{R}^d\times]0,1[)$.

In particular, if $\psi(\cdot,u_n)$ is bounded in $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$, it converges weakly to $\int_0^1\psi(\mathbf{u}(\cdot,\alpha))\,d\alpha$ in $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$.

Since the Itô integration $u\mapsto\int_0^t u\,dw$ is an isometric transformation from $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$ to $\mathcal{M}^2_T(L^2(\mathbb{R}^d))$, the space of all $L^2(\mathbb{R}^d)$-valued continuous, square integrable martingales for the norm of $C([0,T],L^2(\Omega,L^2(\mathbb{R}^d)))$, one gets that $\int_0^t\psi(\cdot,u_n)\,dw$ converges weakly to $\int_0^t\int_0^1\psi(\cdot,\mathbf{u})\,d\alpha\,dw$ in $C([0,T],L^2(\Omega,L^2(\mathbb{R}^d)))$.

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[d]Since, for any set A, $\int_A|\varphi\psi(\cdot,u_n)|\,dt\,dx\,dP\le C(\|\psi(\cdot,u_n)\|_{L^p})[\int_A|\varphi|^q\,dt\,dx\,dP]^{1/q}$.

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