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272

Homogenization Techniques for Composite Media

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PART IV

ELEMENTS OF HOMOGENIZATION FOR INELASTIC SOLID MECHANICS

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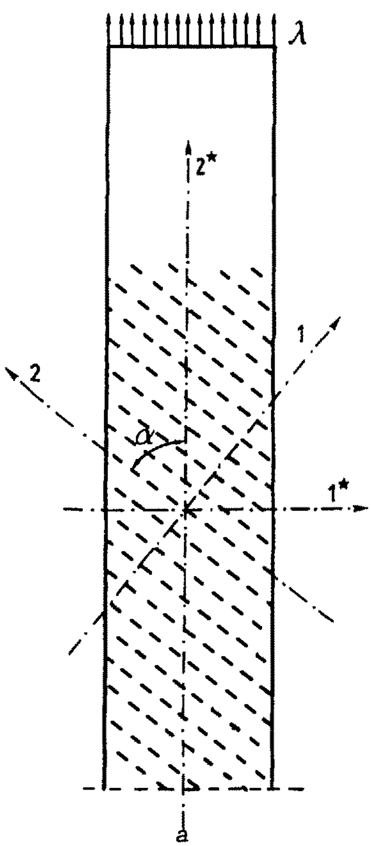
CHAPTER 1 : INTRODUCTION

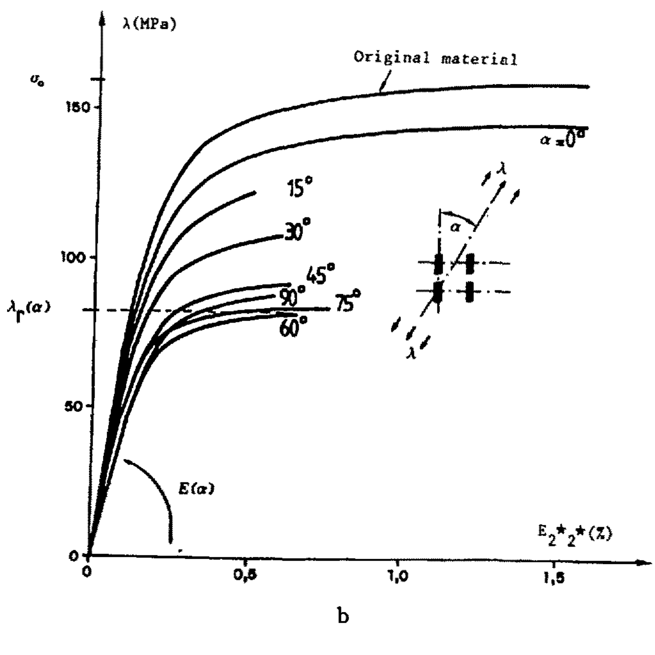
These notes intend to give a brief summary of a few recent developments in the field of the behavior of heterogeneous materials with some emphasis on the dissipative or non-linear range. This topic has been widely discussed in the framework of polycrystals, and the main celebrated contributions by BUDIANSKI & al, HILL, HUTCHINSON, KRONER, MANDEL and others are recalled by A. ZAOUI in this volume. Less attention has been paid to plasticity of composite materials, mainly for two reasons. The first is that most of the composite materials develo ped in the past thirty years exhibit a brittle behavior rather than a ductile one. However, because of the importance of thermal loadings, we have been witnessing a significant development of metal matrix composites, with a highly non linear behavior. The second reason of the limited interest for the non-linear problems is the difficulty of the subject and almost no micro-mechanical problems have yet been solved in a closed form except simple ones. The following HILL's appreciation[1] (1967) is still valid twenty years later : "... As for non-linear systems, the computations needed to establish any complete constitutive law are formidable indeed, even with the piecewise linearization forced by the model". Indeed in most situations we shall limit ourselves to pointing out some simple qualitative facts, or elaborating models based on crude approximations, and we shall often turn to finite element computations to obtain more specific results.

A typical example

Recent experiments by LITEWKA & al [2] illustrate in an illuminating manner the main points in which the present work is interested, and some results of these authors are briefly outlined here. In order to model anisotropic damage they performed tension tests at various inclinations on thin perforated sheets (see figure 1a) . Figure 1b, borrowed from their work, reports the curves external stress/external strain that have been observed at various inclinations.



Tensile tests



stress-strain curves
for various inclinations α

- Figure 1 -

Three different regimes in the behavior of this specific heterogeneous material are evidenced by this figure. For small external stresses and strains the material is in the linearly elastic range. For relatively large strains the external stress reaches a threshold which lead to rupture . A transient part is observed in which the hardening of the original material is affected by the perforations. The present work will devote one section to each of these three typical regimes : linear behavior, rupture of heterogeneous materials, overall elastic plastic hehavior of composites.

Contents

More specifically the paper is organized in the following way :

. Section 2 is devoted to general considerations on representative volume elements (r.v.e.), averaging and micromechanics. We pay a special attention to the boundary conditions imposed on the boundary of the r.v.e. which play an important role in non-linear problems. We set forth the importance and the generality of the so called HILL's macro-homogeneity equality which expresses the principle of virtual work between the microscopic and the macroscopic scales.

. Section 3 is devoted to linear problems. The concept of localization tensors introduced by HILL and MANDEL for heterogeneous elastic materials is exposed. We also consider Maxwell's viscoelastic bodies and we show that short range memory effects for the constituents give rise to long range memory effects for the composite.

. Section 4 is devoted to the failure of heterogeneous materials. We assume that the constituents possess an extremal yield locus which is the limiting set of all physical stress states. We propose a method of constructing the macroscopic extremal yield locus. The proposed set gives an overestimate of the actual set but this estimate turns out to be exact for rigid plastic or elastic plastic constituents.

. In section 5 we discuss the transient part of the stress strain curve of the composite, namely the influence of microscopic elasticity on macroscopic hardening. A large part of the qualitative analysis relies on HILL's and MANDEL's previous works [1,3]. Once the complexity of the exact law is recognized we turn to a few approximate models which yield more quantitative informations.

Notations

Throughout the following Einstein's convention of summation over repeated indices will be adopted. We shall avoid as far as possible the use of indices, denoting by a point or two points the summation over one or two indices. For instance

$\sigma . n$, $\sigma : \varepsilon$, $a : \varepsilon$, $\varepsilon' : a : \varepsilon$ stand for

$\sigma_{ij} n_j$, $\sigma_{ij} \varepsilon_{ji}$, $a_{ijkh} \varepsilon_{hk}$, $\varepsilon'_{ji} a_{ijkh} \varepsilon_{hk}$.

$\mathbb{R}^9_s$ is the space of 3×3 symmetric second order tensors.

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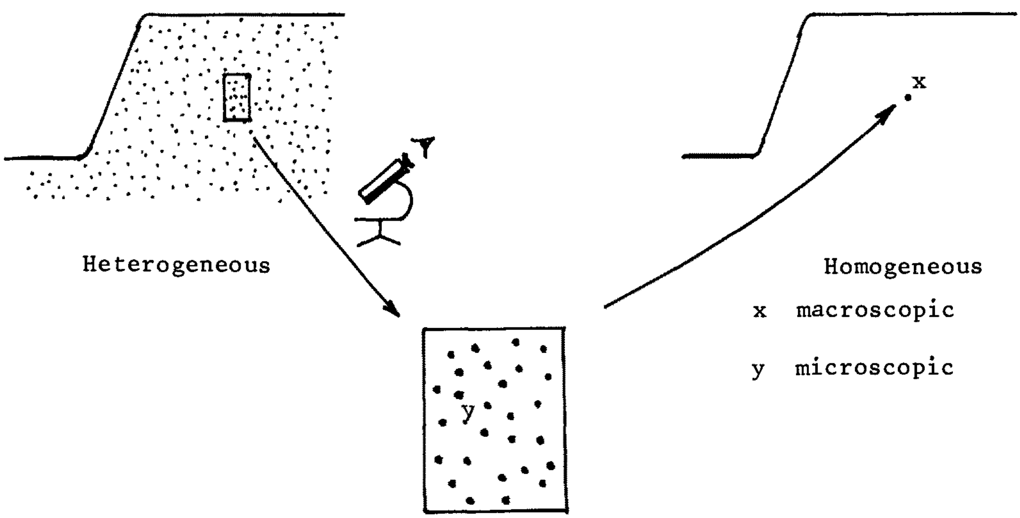
CHAPTER 2

2. AVERAGES. BOUNDARY CONDITIONS

2.1. REPRESENTATIVE VOLUME ELEMENT

In the discussion of the overall properties of a highly heterogeneous medium two different scales are naturally involved :the macroscopic scale (termed x) on which the size of the heterogeneities is very small, and the so called "microscopic" scale (termed y) which is the scale of the heterogeneities. In order to derive a macroscopic (or homogenized) law for the composite one has to assume first that a "statiscally homogeneous specimen" or "representative volume element" can be defined in the composite. Experimentalists know that the assumption of statistical homogeneity can be a difficult matter illustrated for instance by the size effects encountered in the determination of the toughness of a composite. However we will disregard this difficulty and assume that at least one choice of the r.v.e. is possible. This choice of the r.v.e., or its modelling, determines a first difference between various theories of homogenization. In the model of spheres assembly (HASHIN [4]) the r.v.e. is filled with composite spheres of different sizes respecting the volumetric ratios of the phases ; in the self consistent scheme [5] the r.v.e. is successively modelled as an ellipsoidal inclusion of each phases in an infinite matrix endowed with the unknown macroscopic properties. In the homogenization theory of periodic media the r.v.e. is the unit cell, which generates by periodicity the entire structure of the composite. This unit cell is even sometimes modelled by an assembly of parallelepipedic blocks (ABOUDI [6]) . This variety of choices for the r.v.e. eventually results in different expressions of the macroscopic laws but the derivation of the latter follows, most of the time, the general procedure that has been settled by HASHIN [4] , HILL [7] , KRONER [8] and other pioneers of the subject of composite materials.

At a macroscopic point x we must consider two different families of variables : on the one hand macroscopic variables which stand in the homogeneous body the material properties of which we are looking for, on the other hand the microscopic variables which take place in the r.v.e. idealized by x at the macroscopic level.



Representative volume element V

- Figure 2 -

For instance we shall distinguish

Σ $\quad$ E $\quad$ macroscopic stress and strain tensors

and

$\sigma(y)$ $\quad$ $\varepsilon(y)$ $\quad$ microscopic stress and strain tensors.

It results from classical arguments on oscillating functions that the macroscopic stress and strain tensors must be the averages of the microscopic corresponding quantities

$$\left.\begin{aligned} \Sigma_{ij} &= \frac{1}{|V|}\int_V \sigma_{ij}\, dy = \langle \sigma_{ij} \rangle^{(+)} \\ E_{ij} &= \frac{1}{|V|}\int_V \varepsilon_{ij}(u) dy = \langle \varepsilon_{ij}(u) \rangle \end{aligned}\right\} \qquad (1)$$

where $\langle . \rangle$ stands for the averaging operator. However when the heterogeneities are voids or rigid inclusions, the stress or strain tensors remain to be defined in these heterogeneities, and more care is to be applied when considering the equality (1) (cf. § 3.2) .

Moreover, all the mechanical quantities which are usually assumed to be additive functions are averaged when proceeding from the microscopic level to the macroscopic one.

(+) $\langle\ \rangle$ is the average symbol

$$\left.\begin{array}{lll} \bar{\rho} = \langle \rho \rangle & & \text{(additivity of mass)} \\ \bar{\rho}\mathcal{E} = \langle \rho\, e \rangle & & \text{(additivity of internal energy)} \\ \bar{\rho}\, S = \langle \rho\, s \rangle & & \text{(additivity of entropy)} \\ \mathcal{D} = \langle d \rangle & & \text{(additivity of dissipation)} \end{array}\right\} \quad (2)$$

where capital letters refer to Macro quantities ; e , s , d respectively denote the specific internal energy, the specific entropy and the dissipation in the heterogeneous material.

2.2. LOCALIZATION

The procedure which relates Σ , E (and possibly their derivatives with respect to the time and other parameters), by means of (1)(2) , and of the micro constitutive laws, is termed *homogenization*. The inverse procedure, termed *localization*, amounts to a micromechanics problem which permits to determine microscopic quantities, for instance $\sigma(y)$ and $\varepsilon(y)$, from macroscopic ones, Σ and E . For this purpose the following system of equations, with data Σ or E , is to be solved for σ and $\varepsilon(u)$:

$$\left.\begin{array}{l} \text{microscopic constitutive law} \\ \operatorname{div} \sigma = 0 \qquad \text{(micro equilibrium)} \\ \langle \sigma \rangle = \Sigma \quad \text{or} \quad \langle \varepsilon(u) \rangle = E \end{array}\right\} \quad (3)$$

This problem exhibits two noticeable differences with a classical problem:

i) the loading consists in the average value of one field (and not in surface or body forces)

ii) there is no boundary conditions.

<u>Boundary conditions</u>

Problem (3) turns out to be ill posed, due to the absence of boundary conditions, which are to be specified by a careful inspection of the status of the r.v.e. inside the heterogeneous medium. *These boundary conditions must reproduce, as closely as possible, the in situ state of the r.v.e. inside the material.* Therefore they strongly depend on the choice of the r.v.e. itself, and especially on its size. Although the attention will be focused on periodic media, we have to comment on two classical types of boundary conditions (HILL [1,7], HASHIN [4])

$$\text{uniform stresses on } \partial V \; : \; \sigma.n = \Sigma.n \quad \text{on} \quad \partial V \tag{4}$$

or

$$\text{uniform strains on } \partial V \; : \; u = E.y \quad \text{on} \quad \partial V \tag{5}$$

It is immediately seen that a displacement field which satisfies (5), and a divergence free field σ which satisfies (4), also satisfy

$$\langle \varepsilon(u) \rangle = E \quad , \quad \langle \sigma \rangle = \Sigma \quad .$$

In order to justify (4) or (5) (which are *not* equivalent boundary conditions) the r.v.e. must have a large size with respect of the heterogeneities size, so that the stress vector $\sigma.n$ or the displacement u on ∂V fluctuate about a mean with a wavelength small compared with the dimensions of the r.v.e.

However if periodic media are under consideration, and if the r.v.e. is chosen to be the unit cell the fluctuations of these fields about their average are large, and (4) or (5) are to be rejected. For a unit cell located at a sufficiently large distance from the boundary of the heterogeneous body, the strain and stress fields conform at the microscopic level to the periodicity of the geometry : σ and ε are "periodic fields", in a manner which will be specified soon. However it is already clear that the fields σ and ε, which depend on the two variables x (macro) and y (micro) are not exactly periodic throughout the composite : depending on the macrovariable they can vary from one place to the other, in a way similar to that of their averages $\Sigma(x)$ and $E(x)$. However their local variations, taken into account by their dependence on y, are supposed to be periodic. The precise meaning of these periodicity conditions is the following one :

<u>stress</u> : the stress vectors $\sigma.n$ are opposite on opposite sides of ∂V (where the external normal vectors n are also opposite) ;

<u>strain</u> : the local strain $\varepsilon(u)$ is split into its average and a fluctuating term

$$\varepsilon(u) = E + \varepsilon(u^\star) \quad , \quad \langle \varepsilon(u^\star) \rangle = 0 \quad ,$$

E is the Macro-strain, while $u^\star$ can be shown to be a periodic field, up to a rigid displacement that we disregard. The final form of the periodicity conditions on ∂V is :

$$\sigma.n \text{ anti periodic} \quad , \quad u = Ey + u^\star \quad u^\star \text{ periodic.} \qquad (6)$$

We term (4) (5) or (6) a set of "boundary conditions on ∂V" *for the pair* (u,σ) : (4) imposes stringent requirements on σ and none on

u , (5) imposes stringent requirements on u and none on σ , while (6) imposes requirements on both fields. For a specified set of boundary conditions, a displacement field u satisfying the boundary conditions will be said to be an *admissible displacement field*, while a divergence free stress field σ satisfying the boundary conditions will be said to be an *admissible stress field*. If moreover these fields satisfy

$$<\varepsilon(u)> = 0 \quad , \quad \text{or} \quad <\sigma> = 0$$

they are called "purely fluctuating fields" and a purely fluctuating stress field is a *self equilibrated* stress field.

Once the boundary conditions (4) (5) or (6) , are specified, the localization problem (3) is well posed (this assertion is to be checked in details for each constitutive law). In the variational discussion of this problem, the equality of virtual work plays obviously an important role, and can be expressed in simple terms.

Proposition 1. Let $\overline{\sigma}$ *and* $\overline{u}$ *be admissible fields of stress and displacements. Then the average of the microscopic work of* $\overline{\sigma}$ *in the strain field* $\varepsilon(\overline{u})$ *is equal to the macroscopic work* $\overline{\Sigma} : \overline{E}$

$$<\overline{\sigma} : \varepsilon(\overline{u})> = \overline{\Sigma} : \overline{E} \quad . \tag{7}$$

In order to prove (7) for the three sets of boundary conditions we introduce the purely fluctuating parts of $\overline{\sigma}$ and $\varepsilon(\overline{u})$

$$\overline{\sigma} = \overline{\Sigma} + \overline{\sigma}^{\star} \quad , \quad \text{where} \quad <\overline{\sigma}^{\star}> = 0 \quad , \quad \operatorname{div} \overline{\sigma}^{\star} = 0$$

$$\varepsilon(\overline{u}) = \overline{E} + \varepsilon(\overline{u}^{\star}) \quad \text{where} \quad \langle \varepsilon(\overline{u}^{\star}) \rangle = 0 \quad .$$

An easy computation shows that

$$\langle \overline{\sigma} : \varepsilon(\overline{u}) \rangle = \langle (\overline{\Sigma} + \overline{\sigma}^{\star}) : \varepsilon(\overline{u}) \rangle = \overline{\Sigma} : \overline{E} + \langle \overline{\sigma}^{\star} : \varepsilon(\overline{u}) \rangle \quad ,$$

and

$$\langle \overline{\sigma} : \varepsilon(\overline{u}) \rangle = \langle \overline{\sigma} : (\overline{E} + \varepsilon(\overline{u}^{\star})) \rangle = \overline{\Sigma} : \overline{E} + \langle \overline{\sigma} : \varepsilon(\overline{u}^{\star}) \rangle \cdot$$

A proper use of Green's theorem and of the equilibrium equations yields on the one hand

$$\langle \text{micro-work} \rangle = \overline{\Sigma} : \overline{E} + \frac{1}{|V|} \int_{\partial V} \overline{\sigma}^{\star}.n.\overline{u} \, ds \quad , \tag{8}$$

and on the other hand

$$\langle \text{micro-work} \rangle = \overline{\Sigma} : \overline{E} + \frac{1}{|V|} \int_{\partial V} \overline{\sigma}.n.\overline{u}^{\star} \, ds \quad . \tag{9}$$

If $\overline{\sigma}$ satisfies the boundary conditions (4) , i.e. if $\overline{\sigma}.n$ is uniform on ∂V , then $\overline{\sigma}^{\star}.n$ vanishes on ∂V and the equality (7) follows directly from (8) . If $\overline{u}$ satisfies (5) , then $\overline{u}^{\star}$ vanishes on ∂V and (7) follows directly from (9) . If $\overline{\sigma}$ and $\overline{u}$ satisfy (6) , the boundary integral in the second member of (9) vanishes since $\overline{\sigma}.n$ takes opposite values on opposite sides of ∂V , while $\overline{u}^{\star}$ takes equal values on these sets. This ends the proof of (7) which holds true for the three sets of boundary conditions (4) (5) or (6) . We shall term (7) the *equality of virtual work*(+) between the micro-

(+) (7) has sometimes been termed (7) HILL's Macrohomogeneity equality, or HILL's condition.

scopic scale and the macroscopic scale.

Remark. The equality (7) plays a central role in any homogenization theory. Up to a certain extent the boundary conditions on the boundary of the r.v.e. have a minor importance provided that they ensure the validity of (7). However in some statistical theories (7) is interpreted as an ergodic assumption, and deviations from it are sometimes considered (KRONER [8]).

Functional setting

The boundary conditions (4) (5) or (6) and the equilibrium equations can be expressed in a more compact manner, especially convenient for use of variational methods, namely

$$\left.\begin{array}{ll} u = E.y + u^\star \quad , & u^\star \in \mathcal{V}_o \\ \sigma \in S_o = \varepsilon(\mathcal{V}_o)^\perp & \end{array}\right\} \qquad (10)$$

where the space $\mathcal{V}_o$ of fluctuating displacements is one of the following ones, according to the type of selected boundary conditions :

case (4) $\quad \mathcal{V}_o = \hat{\mathcal{V}} = \{u^\star \in H^1(V)^3 \ ; \ \langle \varepsilon(u) \rangle = 0\}$

case (5) $\quad \mathcal{V}_o = \tilde{\mathcal{V}} = \{u^\star \in H^1(V)^3 \ ; \ u = 0 \ \text{ on } \ \partial V\}$

case (6) $\quad \mathcal{V}_o = \mathcal{V}_{per} = \{u^\star \in H^1(V)^3 \ ; \ u \ \text{ periodic on } \ \partial V\}$.

The macroscopic strain associated to a fluctuating $u^\star$ vanishes.

Therefore the equality of virtual work (7) yields

$$\sigma \text{ admissible} \Leftrightarrow \langle \sigma : \varepsilon(u^\star) \rangle = 0 \text{ for every } u^\star \text{ in } V_o$$
$$\text{i.e. } \sigma \in \varepsilon(V_o)^{\perp} \quad .$$

The space of self equilibrated stress fields will be denoted by SE

$$SE = \{\sigma^\star \in \varepsilon(V_o)^{\perp} \; ; \; \langle \sigma^\star \rangle = 0\} \quad .$$

In the next sections it will be understood that a choice of the boundary conditions, i.e. of the space V_o has been made among $\tilde{V}$, $\hat{V}$ or V_{per} .

CHAPTER 3

3. LINEAR PROBLEMS

We apply in this section the above considerations to linear constitutive laws, linear elasticity on the one hand, linear viscoelasticity on the other hand. It is to be understood that a choice of the boundary conditions on ∂V has been made, leading to a choice of V_o .

3.1 LINEAR ELASTICITY

Localization

The localization problem (3) in linear elasticity reads as

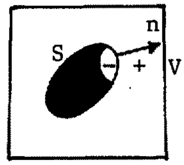
$$\left.\begin{aligned} &\sigma(y) = a(y) : \varepsilon(u(y)) = a(y) : (E + \varepsilon(u^\star(y)) \\ &\text{div } \sigma = 0 \quad , \quad \text{and boundary conditions ,} \end{aligned}\right\} \quad (11)$$

where E *or* Σ is given. Since the material is heterogeneous the 4^{th} order tensor of elastic stiffnesses a depends on the micro variable y . The fluctuating displacement $u^\star$ is therefore solution of the following Navier equations

$$\text{div } (a : \varepsilon(u^\star)) = - \text{div } (a : E) \quad , \quad \text{and boundary conditions.} \quad (12)$$

Assuming that the elasticity tensor a is constant on each constituent, it is readily seen that the second member of (12) reduces to body forces concentrated on the interface between the constituents :

$$\operatorname{div}(a : E) = ([a] : E) . n\, \delta_S$$



- Figure 3 -

where $[a] = a^{+} - a^{-}$, and δ_S denotes the Dirac distribution on S .

It is worth noting that this concentrated loading is completely independent of the type of boundary conditions (4) (5) or (6) which have been selected in order to carry out the localization procedure.

It remains to prove that the problem (12) admits a solution, when Σ or E is given.

Proposition 2 . Under classical assumptions on the elastic tensor a *the problem* ((2) *admits a unique solution* $(\sigma, \varepsilon(u))$ *whatever is the set of boundary conditions* (4) (5) *or* (6) .

<u>E given</u>

Taking advantage of the fact that $<\sigma : \varepsilon(v)>$ vanishes for every v in V_o we obtain the following variational formulation of (12)

$$\left.\begin{array}{l} u^{\star} \in V_o \\ <\varepsilon(v) : a : \varepsilon(u^{\star})> = -<\varepsilon(v) : a : E> \quad \text{for every } v \text{ in } V_o \end{array}\right\} (13)$$

It can be proved that $\varepsilon(V_o)$ is a Hilbert space (Problem 3.1), when endowed with the scalar product $\langle \varepsilon : \varepsilon' \rangle$ ($\varepsilon, \varepsilon'$ belong to $\varepsilon(V_o)$). Then, under the classical assumptions of symmetry coercivity and boundedness of the elasticity tensor, the bilinear form

$$(\varepsilon, \varepsilon') \longrightarrow \langle \varepsilon : a : \varepsilon' \rangle$$

is symmetric, continuous and coercive on the Hilbert space $\varepsilon(V_o)$. In a similar manner the following linear form is continuous on $\varepsilon(V_o)$

$$\varepsilon \longrightarrow \langle \varepsilon : a : E \rangle$$

Thus, LAX-MILGRAM's theorem ensures existence and uniqueness of a solution $\varepsilon^\star = \varepsilon(u^\star)$, $u^\star$ in V_o, for (13). Existence and uniqueness of $\varepsilon(u) = \varepsilon^\star + E$ and of $\sigma = a : \varepsilon(u)$ follow directly.

Since the problem (13) is linear, its solution $\varepsilon(u^\star)$ depends linearly on the data E. More specifically let I_{ij} denotes the 2^{nd} order tensor with components

$$(I_{ij})_{kh} = \frac{1}{2}(\delta_{ik}\,\delta_{jh} + \delta_{ih}\,\delta_{jk}) \quad ,$$

I the identity 4^{th} order tensor has components

$$(I)_{ijkh} = (I_{ij})_{kh} \quad ,$$

and let $\varepsilon(\chi_{kh})$ denotes the solution of (13) when $E = I_{kh}$. $\varepsilon(\chi_{kh})$ is the field of fluctuating strains induced at the microscopic level by the 6 elementary states of macroscopic strain :

$$I_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad 2\,I_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \cdots\cdots$$

(extension in direction 1)   (shear strain between directions 1 and 2)

The solution $\varepsilon(u^{\star})$ of (13) for a general macro strain E is the superposition of the elementary solutions $\varepsilon(\chi_{kh})$

$$\varepsilon(u^{\star}) = E_{kh}\ \varepsilon(\chi_{kh}) \tag{14}$$

Finally the total field of microstrains amounts to

$$\varepsilon(u) = E_{kh}(I_{kh} + \varepsilon(\chi_{kh}))$$

$$\text{i.e.} \quad \varepsilon_{ij}(u) = D_{ijkh}\ E_{kh} = (D : E)_{ij} \tag{15}$$

where $D_{ijkh} = ((I_{kh})_{ij} + \varepsilon_{ij}(\chi_{kh}))$

D is 4^{th} order tensor of *strain localization*(+) , since it yields the local strain $\varepsilon(u)$ in terms of the macroscopic strain E .

Homogenization

Once the localization procedure is known by (15) , the homogenization itself is straightforward :

(+) also termed "influence tensor" (HILL [1]), or "concentration tensor" (MANDEL [3])

$$\Sigma = \langle\sigma\rangle = \langle a : \varepsilon(u)\rangle = \langle a : D : E\rangle = \langle a : D\rangle : E$$

Then $\Sigma = a^{hom} : E$ where $a^{hom} = \langle a : D\rangle$.

In order to prove the symmetry of a^{hom} , which does not appear clearly on the above expression, we note further properties of the localization tensor D .

$$\langle D\rangle = I \ , \quad \langle D^T\rangle = I \ ,$$

and for every admissible stress field $\overline{\sigma}$

$$\langle D^T : \overline{\sigma}\rangle_{ij} = \langle D^T_{ijkh}\ \overline{\sigma}_{kh}\rangle = \langle [(I_{ij})_{kh} + \varepsilon_{kh}(\chi_{ij})]\overline{\sigma}_{kh}\rangle = \overline{\Sigma}_{ij}$$

$$\text{i.e.} \quad \langle D^T : \overline{\sigma}\rangle = \overline{\Sigma} \quad \text{for every admissible } \overline{\sigma} \tag{16}$$

The equality (16) allows to derive an equivalent expression of a^{hom} :

$$\Sigma = \langle D^T : \sigma\rangle = \langle D^T : a : \varepsilon(u)\rangle = \langle D^T : a : D\rangle : E$$

$$\text{i.e.} \quad a^{hom} = \langle D^T : a : D\rangle \tag{17}$$

This last equality, which clearly shows the symmetry of a^{hom} , can be derived by energy considerations. Let us apply the averaging process to internal energies (cf. (2)) :

$$\overline{\rho}\mathscr{E} = \frac{1}{2} E : a^{hom} : E = \langle\rho e\rangle = \langle\frac{1}{2}\, \varepsilon(u) : a : \varepsilon(u)\rangle$$

$$= \frac{1}{2} E : \langle D^T : a : D\rangle : E \ ,$$

which yields the expression (17) for a^{hom} .

Σ given

We now consider the localization problem (11) when Σ is given :

$$\left.\begin{array}{ll} \varepsilon(u) = \varepsilon(u^{\star}) + E = A : \sigma & (E \text{ is unknown}) \\ \operatorname{div} \sigma = 0 \text{ , and boundary conditions} & \\ \langle\sigma\rangle = \Sigma & (\Sigma \text{ is known}) \end{array}\right\} \quad (18)$$

where A is the 4^{th} order tensor of elastic compliances, inverse of a (A depends on the microscopic variable y) . Admit for a moment that (18) has a unique solution σ . Since the problem is linear its solution depends linearly on the data Σ . More specifically if C_{kh} denotes the solution of (18) with $\Sigma = I_{kh}$, we have

$$\sigma(y) = \Sigma_{kh}\, C_{kh}(y)$$

Let us define the 4^{th} order tensor of *stress localization* C , by

$$C_{ijkh} = (C_{kh})_{ij} \text{ ,}$$

then $\sigma = C : \Sigma$ (19)

This last equality allows to compute the tensor of macroscopic elastic compliances

$$E = \langle\varepsilon(u)\rangle = \langle A : \sigma\rangle = \langle A : C\rangle : \Sigma$$

i.e. $\quad A^{hom} = \langle A : C \rangle \quad .$ (20)

Further properties of C^T help to prove the symmetry of A^{hom} : $\langle C^T \rangle = I$, and for every admissible strain field $\varepsilon(\bar{u})$

$$\langle C^T : \varepsilon(\bar{u}) \rangle_{ij} = \langle C^T_{ijkh} \, \varepsilon_{kh}(\bar{u}) \rangle = \langle (C_{ij})_{kh} \, \varepsilon_{kh}(\bar{u}) \rangle$$

by the equality of virtual work

$$= \langle (C_{ij})_{kh} \rangle \langle \varepsilon_{kh}(u) \rangle = \bar{E}_{ij} .$$

Therefore

$$E = \langle C^T : \varepsilon(u) \rangle = \langle C^T : A : \sigma \rangle = \langle C^T : A : C \rangle : \Sigma$$

and $\quad A^{hom} = \langle C^T : A : C \rangle$ (21)

It remains to prove the existence and uniqueness of a solution of (18) . Taking advantage of the fact that $\langle \tau : \varepsilon(u) \rangle$ vanishes for every self equilibrated stress field τ , we obtain the following variational formulation of (18)

$$\left. \begin{array}{l} \sigma \in K \\ \\ \langle \bar{\sigma} - \sigma : A : \sigma \rangle = 0 \quad \text{for every } \bar{\sigma} \text{ in } K \end{array} \right\} \qquad (22)$$

where $K = \{\bar{\sigma} \in \varepsilon(V_o)^{\perp} \; ; \; \langle \bar{\sigma} \rangle = \Sigma\}$.

Under the classical assumptions of symmetry, coercivity and boundedness of A , the bilinear form

$$(\tau,\sigma) \rightarrow \langle \tau : A : \sigma \rangle$$

is symmetric, continuous and coercive on $L^2(V)^9_s$. K is clearly a non empty closed convex set of this space and LAX-MILGRAM's theorem ensures the existence and uniqueness of a solution σ of (22) . Existence and uniqueness of $\varepsilon(u) = A : \sigma$ follows directly. This ends the proof of proposition 2 .

Note that we can write (22) in an equivalent form

$$\left.\begin{array}{l} \sigma \in K \\ \\ \langle \tau : A : \sigma \rangle = 0 \quad \text{for every } \tau \text{ in } SE \end{array}\right\} \qquad (23)$$

<u>Equivalence between imposed strains and imposed stresses</u>

The tensors a^{hom} and A^{hom} constructed by imposing either a given E , or a given Σ are inverse tensors, provided that the *same* boundary conditions (4) (5) or (6) have been chosen to solve the localization problems. Indeed using the symmetry of a^{hom} we get

$$a^{hom} : A^{hom} = a^{hom\ T} : A^{hom} = \langle D^T : a \rangle : \langle A : C \rangle$$

But we notice from the true definition of D and C that $D^T : a$ is an admissible stress field

$$(D^T : a)_{ijkh} = a_{pqkh}[(I_{ij})_{pq} + \varepsilon_{pq}(\chi_{ij})] \quad ,$$

while $A : C$ is an admissible strain field

$$(A : C)_{khlm} = A_{khrs}(C_{lm})_{rs} \, .$$

Then by the equality of virtual work

$$<D^T : a> : <A : C> = <D^T : a : A : C> = <D^T : C>$$

$$= <D^T> : <C> = I \, .$$

Once the boundary conditions have been chosen among (4) (5) or (6) or any other type (see Problems), we can compute several pairs of elasticity tensors

$$\begin{array}{lll} \mathcal{V}_o = \hat{\mathcal{V}} & \hat{a}^{hom} , \hat{A}^{hom} & \\ \mathcal{V} = \mathcal{V}_{per} & a^{hom}_{per} , A^{hom}_{per} & \qquad (24) \\ \mathcal{V} = \tilde{\mathcal{V}} & \tilde{a}^{hom} , \tilde{A}^{hom} & . \end{array}$$

As it has been proved previously the compliances tensors and the stiffnesses tensors computed by the same type of boundary conditions are inverse. However we point out that constructing the stiffnesses tensor by the assumption of uniform strain on ∂V , and the compliances tensor by the assumption of uniform stress on ∂V leads to an approximate theory, since these two tensors are not rigourously inverse. As pointed out by HILL [1] and MANDEL [3]

$$\tilde{a}^{hom} : \hat{A}^{hom} - I = 0((\frac{d}{\ell})^3)$$

where d is the typical size of the heterogeneities, and ℓ is the typical size of the r.v.e. . For large r.v.e., containing of large number of heterogeneities the ratio d/ℓ is small, and the choice of the boundary conditions is unimportant. However for periodic media, when the r.v.e. is taken to be a unit cell, d and ℓ are of the same order, the different boundary conditions lead to substantial differences.

It can be proved (cf. Problems) that the strain energies defined by the 3 tensors (24) are ordered in the following way

$$E : \hat{a}^{hom} : E \leqslant E : a^{hom}_{per} : E \leqslant E : \tilde{a}^{hom} : E$$

(reverse inequalities for compliances).

For a periodic medium, the assumption of uniform strains on ∂V overestimates the stiffnesses, while the assumption of uniform stresses on ∂V underestimates it.

3.2 LINEAR ELASTICITY. COMPARISON EXPERIMENTS/COMPUTATIONS

We turn back to the tensile tests on perforated thin sheets, performed by LITEWKA & al [2] as described in the introduction. In view of the periodicity of the structure the r.v.e. is chosen to be the unit cell. The boundary conditions are of the periodic type (6) and $\mathcal{V}_o$ equals $\mathcal{V}_{per}$. Since a solution in a closed form seems to be unattainable, we solve the localization problem (13) (imposed macroscopic strain) by a finite element method. The computations are performed under the plane stress assumption and the only elementary macroscopic strains which are of interest here, are

$$I_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad I_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad 2I_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since the heterogeneities considered in this example are voids we need to comment the definition of the macroscopic strain and stress. Indeed the expressions Σ and E are ambiguous since σ and ε are not defined in the hole. However assuming that the void consists in a infinitely soft heterogeneity, we can extend the fields σ and u , into $\bar{\sigma}$ (+) and $\bar{u}$ everywhere defined on the r.v.e. . Thus

---

(+) $\bar{\sigma}$ clearly vanishes in the void

$$\Sigma_{ij} = \langle \overline{\sigma}_{ij} \rangle = \frac{1}{|V|} \int_V \overline{\sigma}_{ij}\, dy = \frac{1}{|V|} \int_{V^\star} \sigma_{ij}\, dy$$

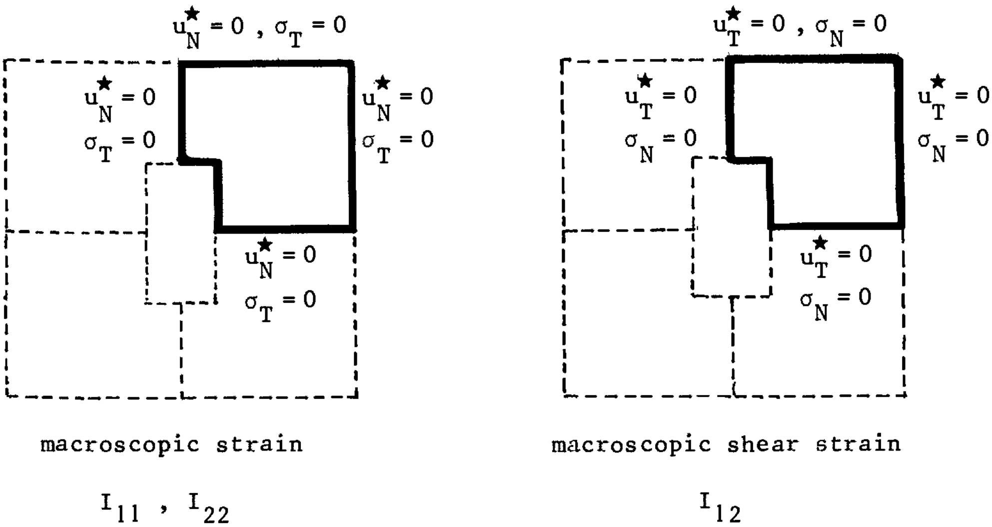
$$E_{ij} = \langle \varepsilon_{ij}(\overline{u}) \rangle = \frac{1}{|V|} \int_V \varepsilon_{ij}(\overline{u})\, dy = \frac{1}{|V|} \int_{\partial V} \frac{1}{2}(\overline{u}_i n_j + \overline{u}_j n_i)\, ds$$

$$= \frac{1}{|V|} \int_{\partial V} \frac{1}{2}(u_i n_j + u_j n_i)\, ds$$

where V denotes the r.v.e., including the hole T, $V^\star$ denotes the material part of V, $V^\star = V - T$, and where we have assumed that T does not intersect ∂V (hence $\overline{u} = u$ on ∂V). With this modified definition of Σ and E, the whole preceeding section remains valid.

Periodicity conditions

The finite element computations required to solve (13) with $E = I_{11}$, I_{22}, I_{12} are standard, except for the periodicity conditions. However these periodicity conditions reduce to ordinary ones if the unit cell admits two orthogonal axis of symmetry. In the examples under consideration here, the unit cells are symmetric with respect to the lines $y_1 = 0$ and $y_2 = 0$. It is easily shown that the periodicity boundary conditions reduce to usual ones indicated on figure 4 below, and that the computations can be carried out on a quarter cell



\- Figure 4 -

However in two important situations we cannot get rid of these periodicity conditions :

a) when the unit cell has no axis of symmetry. In this connection DUVAUT [9] and coworkers studied the influence of the shape of fibers cross section on the macroscopic stiffness of unidirectional composites, and the reader is referred to this work for more details.

b) when non linear materials are under consideration, the superposition principle (14) no more applies and it is not possible to separate the extension part and the shear part of a general macroscopic strain E , in order to carry out the numerical computations.

A survey of a few direct numerical procedures accounting for the periodicity conditions is given in DEBORDES & al [10] and MARIGO & al [11] .

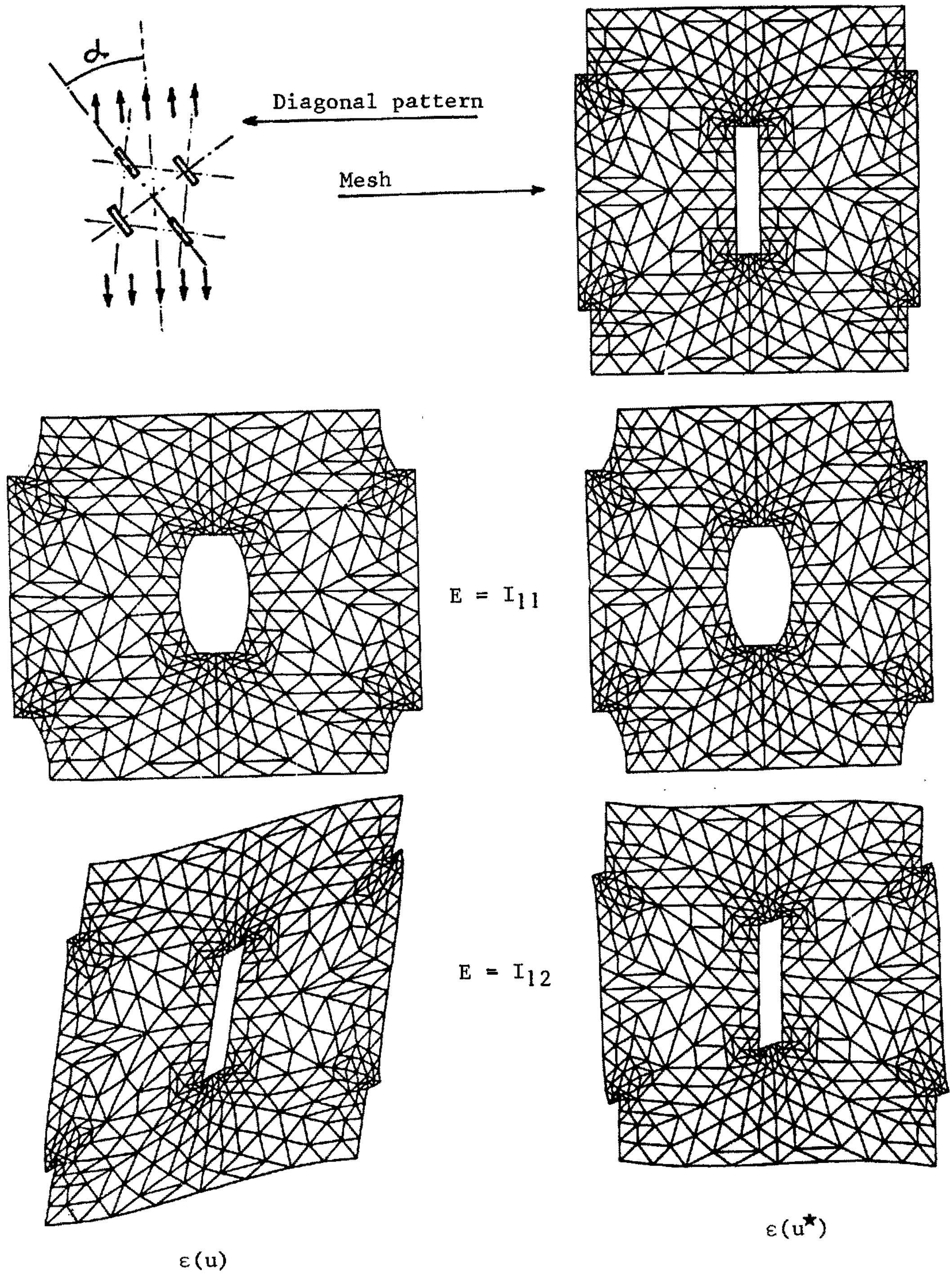
Diagonal perforation pattern

Figures 5 & 6 show a few numerically computed microscopic strain, or stress states. The deformation of the perforation, and the stress concentration on its boundary are evidenced. A few noticeable facts deserve brief comments :

a) the deformation of the perforation is mainly due to the fluctuating strain $\varepsilon(\chi)$ which accounts for the heterogeneity of the material.

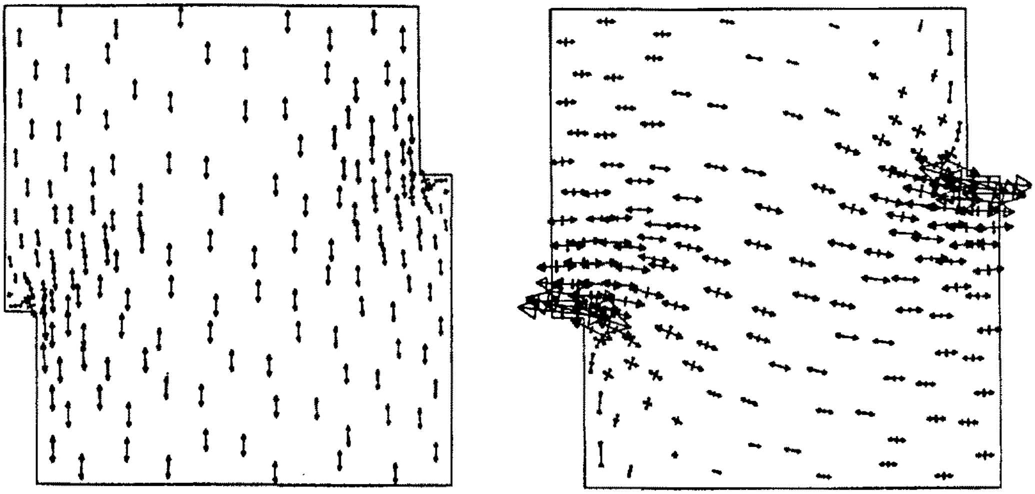
b) the deformed state corresponding to a shear macroscopic strain I_{12} clearly shows that the strain is *not uniform* on ∂V since straight lines do not remain straight lines. Indeed in most of the computations that the author has performed, the most significant differences between the various types of boundary conditions were observed on macroscopic strains of *shear* (with respect to the axis of the unit cell), and these differences resulted in significant variations of the shear moduli.

c) the agreement between the computations performed with the periodicity conditions and experiments is quite satisfactory.



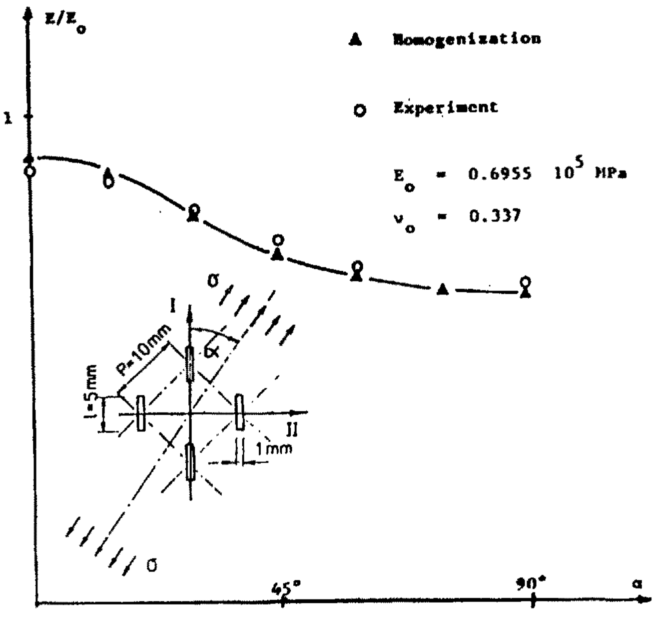
Diagonal pattern
Mesh
α
$E = I_{11}$
$E = I_{12}$
$\varepsilon(u)$
$\varepsilon(u^*)$

\- Figure 5 -



$$\Sigma = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \qquad\qquad \Sigma = \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$$

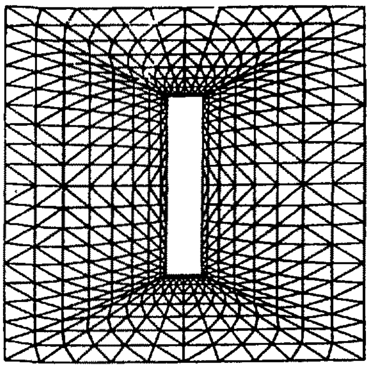
microscopic stresses on a quarter cell



- Figure 6 -

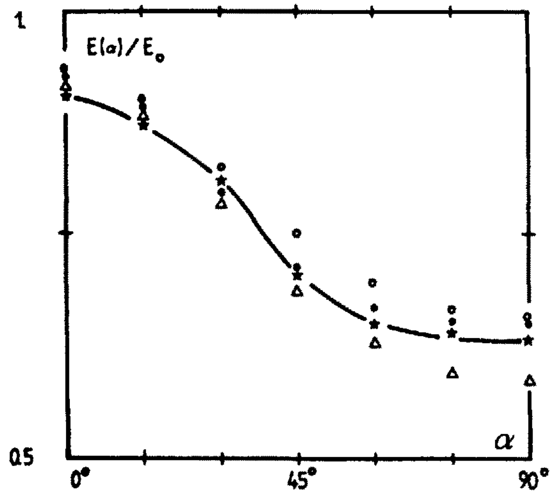
Square perforation pattern (+)

The three types of boundary conditions (4) (5) or (6) are compared on the square perforation pattern. The stiffnesses predicted by the 3 theories are ordered in the way that energy arguments indicate : overestimate by the uniform strain theory, underestimate by the uniform stress theory.



Mesh

Square pattern



o Uniform strains on ∂V

● Periodicity conditions

★ Experiment [2]

Δ Uniform stresses on ∂V

Stiffnesses/orientation

\- Figure 7 -

(+) The numerical computations for the square perforation pattern have been performed by J.C. MICHEL [12] .

3.3 LINEAR VISCOELASTICITY

We consider an assembly of viscoelastic constituents obeying the MAXWELL's law :

$$\varepsilon(\dot{u}(y)) = A(y) : \dot{\sigma}(y) + B(y) : \sigma(y) \tag{25}$$

where A and B are 4^{th} order tensors endowed with all the recommended properties (symmetry, boundedness, coercivity). Maxwell's law (25) has a "short memory" since it does not involve any creep function accounting for long range memory effects. The main result of this section is the following one.

Proposition 3 . The macroscopic law derived from (25) *is*

$$\dot{E} = A^{hom} : \dot{\Sigma} + \int_0^t J(t-s) : \dot{\Sigma}(s)ds + B^{hom} : \Sigma \tag{26}$$

where the kernel J *will be specified in the text.*

Therefore the homogenization procedure gives rise to *long memory effects*, characterized by J . A similar result for KELVIN-VOIGT's materials was pointed out earlier by SANCHEZ & al [13] and FRANCFORT & al [14] .

Localization

As in the elastic case the main point of the homogenization procedure is the localization problem. We assume that the macroscopic stress tensor follows a given path $\Sigma(t)$ and we search for the microscopic state $\sigma(t)$, $\varepsilon(u(t))$ thereby induced. The localization problem amounts to

$$\left.\begin{array}{l} A : \dot{\sigma} + B : \sigma = \varepsilon(\dot{u}) = \dot{E} + \varepsilon(\dot{u}^{\star}) \\ \operatorname{div} \sigma = 0 \quad , \quad \text{and boundary conditions} \\ \langle \sigma(t) \rangle = \Sigma(t) \quad . \end{array}\right\} \qquad (26)$$

Applying the Laplace transform to (26) yields

$$\left.\begin{array}{l} (\lambda A + B)\hat{\sigma}(\lambda) = \hat{\dot{E}}(\lambda) + \varepsilon(\hat{\dot{u}}^{\star}(\lambda)) \\ \operatorname{div} \hat{\sigma}(\lambda) = 0 \quad , \quad \text{and boundary conditions} \\ \langle \hat{\sigma}(\lambda) \rangle = \hat{\Sigma}(\lambda) \quad , \end{array}\right\} \qquad (27)$$

where $\sigma(t)$ is taken as vanishing in $t = 0$. Therefore we are let with a localization problem for a fictitious elastic material with elastic compliance $\lambda A + B$. Let us denote by $C(\lambda)$ the stress localization for the latter operator, solution of the following variational problem

$$\left.\begin{array}{l} \langle \tau : (A + \frac{1}{\lambda} B) : C(\lambda) \rangle = 0 \quad \text{for every } \tau \text{ in } SE \\ \langle C(\lambda) \rangle = I \quad , \quad C(\lambda) \in \varepsilon(V_o)^{\perp} \quad . \end{array}\right\} \qquad (28)$$

Let us denote by C^A and C^B the elastic stress localization tensors associated with A and B respectively, solutions of the following variational problem

$$\left.\begin{array}{l} \langle C^B \rangle = I \quad , \quad C^B \in \varepsilon(V_o)^{\perp} \\ \\ \langle \tau : B : C^B \rangle = 0 \quad \text{for every } \tau \text{ in } SE \end{array}\right\} \qquad (29)$$

(similar problem for C^A) .

We set $C^{\star}(\lambda) = (C(\lambda) - C^B)/\lambda$ and we note that $C^{\star}(\lambda)$ satisfies

$$C^{\star}(\lambda) \in SE$$
$$\langle \tau : \lambda A + B : C^{\star}(\lambda) \rangle + \langle \tau : A : C^B \rangle = 0 \quad \text{for every } \tau \in SE \quad .$$

Using (29) for C^A yields another expression of the second term of the above equality :

$$\langle \tau : A : C^B \rangle = \langle \tau : A : C^B - C^A \rangle \quad ,$$

and $C^{\star}(\lambda)$ satisfies

$$C^{\star}(\lambda) \in SE \text{ , and for every } \tau \text{ in } SE$$
$$\langle \tau : A : (\lambda C^{\star}(\lambda) - C^A + C^B) \rangle + \langle \tau : B : C^{\star}(\lambda) \rangle = 0 \quad .$$

It is readily seen that $C^{\star}(\lambda)$ is the Laplace transform of the fourth order tensor $C(t)$ solution of the following evolution equation

$$\left.\begin{array}{l} C(t) \in SE \quad , \quad C(0) = C^A - C^B \\ \\ \langle \tau : A : \dot{C} \rangle + \langle \tau : B : C \rangle = 0 \quad \text{for every } \tau \text{ in } SE \end{array}\right\} \qquad (30)$$

which admits a unique solution $C(y,t)$. Coming back to $C(\lambda)$ we see that

$$C(\lambda) = \lambda C^{\star}(\lambda) + C^{B}$$

and

$$\hat{\sigma}(\lambda) = C(\lambda) : \hat{\Sigma} = C^{\star}(\lambda) : \lambda\hat{\Sigma} + C^{B} : \hat{\Sigma} .$$

Applying the inverse Laplace transform[(+)] yields

$$\sigma(t) = \int_0^t C(t-s) : \dot{\Sigma}(s)ds + C^{B} : \Sigma(t) .$$

This completes the localization procedure.

Homogenization

Noting that

$$\dot{\sigma}(t) = C^{A} : \dot{\Sigma}(t) + \int_0^t \dot{C}(t-s) : \dot{\Sigma}(s)ds$$

we obtain

$$\varepsilon(\dot{u}) = A : C^{A} : \dot{\Sigma} + \int_0^t [A : \dot{C}(t-s) + B : C(t-s)] : \dot{\Sigma}(s)ds$$

$$+ B : C^{B} : \Sigma(t)$$

where A , C^{A} , B , C^{B} and C depends on the microscopic variable y . Averaging gives

---

(+) Note that $\Sigma(t) = 0$ since $\sigma(t) = 0$.

$$\dot{E} = A^{hom} : \dot{\Sigma} + \int_0^t J(t-s) : \dot{\Sigma}(s)ds + B^{hom} : \Sigma(t) \tag{31}$$

where $J(\xi) = \langle A : \dot{C}(\xi) + B : C(\xi) \rangle$.

This completes the proof of proposition 3 .

CHAPTER 4

4. FAILURE OF DUCTILE HETEROGENEOUS MATERIALS

4.1 EXTREMAL YIELDING SURFACE

We assume that each constituent of the composite has an *extremal surface* which delimits the set $P(y)$ of all stress states that the material can physically admit

$$\sigma(y) \in P(y) \qquad y \in V \tag{32}$$

The behavior of the material is not further specified, the only useful information being the constraint (32). In most examples $P(y)$ is defined by means of a yield function $f(y,\sigma)$:

$$P(y) = \{\sigma \mid f(y,\sigma) \leq 0\}$$

Since the microscopic stress field is constrained its average the macroscopic stress has to be constrained too. More specifically let us assume that the yield locus P is defined by means of a (semi) norm $\| \ \|$

$$P(y) = \{\sigma \mid \|\sigma\| \leq \sigma_o(y)\} \quad .$$

Then

$$\|\Sigma\| \leqslant \langle\|\sigma\|\rangle \leqslant \langle\sigma_o\rangle \tag{33}$$

For instance if the norm under consideration is that of the equivalent stress

$$\|\sigma\| = \sigma_{eq} = (\frac{3}{2}\,\sigma^D_{ij}\,\sigma^D_{ij})^{1/2} \quad ,$$

then (33) amounts to

$$\Sigma_{eq} \leqslant \langle\sigma_o\rangle \tag{34}$$

(33) provides a crude but simple upper bound for the macroscopic extremal yield locus, which remains to be defined in a more specific way.

For this purpose we note that, in order that a macroscopic stress Σ can be physically attained it must be possible to find a microscopic stress field σ fulfilling the following requirements :

i) $\langle\sigma\rangle = \Sigma$

ii) $\operatorname{div}\sigma = 0$ and boundary conditions.

Note that i) ii) express that σ is in equilibrium with Σ .

iii) $\sigma(y) \in P(y)$ for every y in V .

It is therefore natural to consider the following set of macroscopic stresses :

$$P^{hom} = \{\Sigma \in \mathbb{R}^9_s \text{ such that there exists } \sigma \text{ satisfying}$$

$$\langle\sigma\rangle = \Sigma \ , \quad \sigma \in \varepsilon(V_o)^{\perp} \ , \quad \sigma(y) \in P(y) \quad \text{for every } y \text{ in } V\} \tag{35}$$

Let us now assume that $P(y)$ exhibits further properties :

i) $P(y)$ is a closed convex set in $\mathbb{R}_s^9$. Then simple arguments show that P^{hom} is a closed convex set in $\mathbb{R}_s^9$. *Convexity is a stable property under homogenization.*

ii) For every y in V $P(y)$ contains a fixed ball of center 0 and of radius $k_o > 0$. Then P^{hom} is a non-empty set since it contains the ball of radius k_o and of center 0 .

Having shown that all physical macroscopic stress states Σ must lie within P^{hom} , the question arise whether all states Σ in P^{hom} are physical macroscopic stress states : the microscopic stress field associated with Σ should be related to a microscopic admissible strain field by the local constitutive law. If we do not further specify the constitutive law, the answer to the question is no : in the vocabulary of SALENCON [15] P^{hom} is the set of "potentially safe" Σ , and not of safe Σ . However if we consider elastic plastic constituents obeying the normality rule, it can be proved through rather technical functional analysis arguments that all stress states in the interior of P^{hom} can be attained. On the contrary,computing P^{hom} for elastic brittle constituents,as are most of the fibers in composite materials, could lead to a serious overestimate of the strength of the composite (see in this connection WEILL [16]) . Therefore the computation of P^{hom} will give a reliable prediction of the failure of a composite materials, only if the constituents are elastic plastic (or rigid plastic).

Throughout the following it will be assumed that $P(y)$ *is a closed convex set and that the constituents obey the normality rule.*

Rigid plastic constituents

Assume that the local constituents are rigid plastic and obey the normality rule. The inequality of maximal plastic work at the microscopic level, is valid for every y in V and reads as

$$\left.\begin{array}{l} \sigma(y) \in P(y) \\ \varepsilon(\dot{u}(y)) : \overline{\sigma} - \sigma(y) \leqslant 0 \quad \text{for every } \overline{\sigma} \text{ in } P(y) \end{array}\right\} \qquad (36)$$

Let $\overline{\Sigma}$ be an element of P^{hom} to which corresponds $\overline{\sigma}(y)$ at the microscopic level by (35) . Then averaging (36) and applying the equality (7) of virtual work yields

$$\left.\begin{array}{l} \Sigma \in P^{hom} \\ \dot{E} : \overline{\Sigma} - \Sigma \leqslant 0 \quad \text{for every } \overline{\Sigma} \text{ in } P^{hom} \end{array}\right\} \qquad (37)$$

If Σ is in the interior of P^{hom} we can take $\overline{\Sigma}$ in the form

$$\overline{\Sigma} = \Sigma + \Sigma^{\star}$$

where $\Sigma^{\star}$ is any vector in $\mathbb{R}^9_s$ with a sufficiently small norm, such that $\overline{\Sigma}$ lies in P^{hom} . Then (37) yields

$$\dot{E} : \Sigma^{\star} \leqslant 0$$

for every $\Sigma^{\star}$ with a sufficiently small norm. This last inequality applied to $\pm\Sigma^{\star}$ turns out to be an equality, and to be valid for every $\Sigma^{\star}$ (multiply it by any scalar value).

Thus $\dot{E} = 0$, and the composite is rigid if Σ is inside P^{hom} . The only possibility of straining occurs when Σ is on the boundary of P^{hom} .

Therefore the composite is rigid plastic, its domain of admissible stresses is exactly P^{hom} *, and it obeys the normality rule.*

4.2 DETERMINATION OF THE EXTREMAL SURFACE

It follows from its true definition (35) , that the determination of P^{hom} amounts to the resolution of a limit analysis problem on the r.v.e., where the loading parameters are the components of Σ . Classically this limit analysis problem can be solved either by the inside, through the construction of statically and plastically admissible fields, or by the outside through the evaluation of the plastic energy rate dissipated in strain fields leading to ruin.

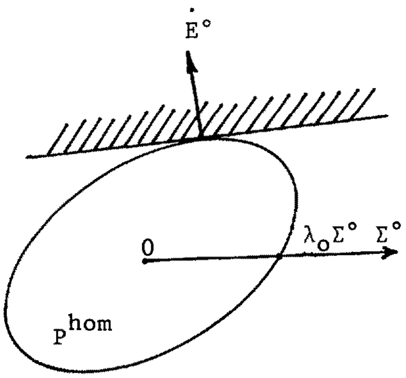
Determination by the inside

A direction Σ° of macroscopic stresses is fixed and we consider the following onedimensional limit analysis problem

$$\lambda_o = \sup \{\lambda \text{ such that there exists } \sigma \text{ satisfying}$$

$$\langle\sigma\rangle = \lambda\Sigma^\circ , \quad \sigma \in \varepsilon(V_o)^{\perp} , \quad \sigma(y) \in P(y) \text{ for every } y \text{ in } V\} \tag{38}$$

$\lambda_o\Sigma^\circ$ is on the boundary of P^{hom} .



- Figure 8 -

Determination by the outside

Let us consider a macroscopic strain rate $\dot{E}$. Then P^{hom} is the intersection of the following half spaces

$$H(\dot{E}) = \{\overline{\Sigma} \mid \overline{\Sigma} : \dot{E} \leqslant \mathcal{D}(\dot{E})\} \tag{39}$$

where $\mathcal{D}$ denotes the energy-rate plastically dissipated at the macroscopic scale in the strain rate $\dot{E}$. $\mathcal{D}$ is computed by means of the averaging procedure :

$$\mathcal{D}(\dot{E}) = \langle d(y,\varepsilon(\dot{u}))\rangle = \inf_{\overline{u} - \dot{E}y \in \mathcal{V}_o} \langle d(y,\varepsilon(\overline{u})\rangle \tag{40}$$

where $d(y,e) = \sup\limits_{\overline{\sigma} \in P(y)} \overline{\sigma} : e$

Indeed we shall prove the following inclusion

$$P^{hom} \subset \bigcap_{\dot{E} \in \mathbb{R}^9_s} H(\dot{E}) \qquad (41)$$

letting the proof of the reverse inclusion to the reader. Let Σ be an element of P^{hom} and σ one possible microscopic stress field fulfilling the requirements of (35) . $\dot{E}$ being given, let $\overline{u}$ be any admissible displacement rate satisfying

$$\overline{u}^{\star} = \overline{u} - \dot{E}y \in V_o \qquad (42)$$

Then by the equality (7) of virtual work

$$\Sigma : \dot{E} = \langle \sigma : \varepsilon(\overline{u}) \rangle \leqslant \langle \sup_{\overline{\sigma} \in P(y)} \overline{\sigma} : \varepsilon(\overline{u}) \rangle = \langle d(y, \varepsilon(\overline{u})) \rangle$$

Taking the infimum over all admissible displacement rates $\overline{u}$ satisfying (42) yields

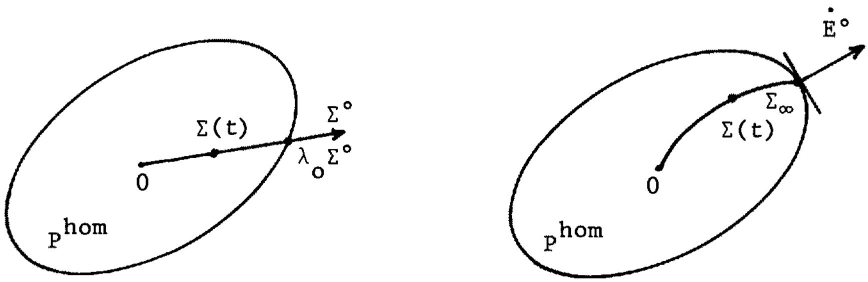
$$\Sigma : \dot{E} \leqslant \mathcal{D}(\dot{E}) \quad \text{for every} \quad \dot{E} \in \mathbb{R}^9_s$$

which proves that Σ belongs to $H(\dot{E})$ for every $\dot{E}$ in $\mathbb{R}^9_s$.

Numerical determination of P^{hom} .

In order to numerically determine P^{hom} we solve an auxilliary evolution problem for a fictitious elastic perfectly plastic material obeying the normality rule and admitting $P(y)$ as its local yield locus.

Two types of loadings can be considered. Either a direction of macroscopic stress $\Sigma°$ is specified,and the evolution problem yields asymptotically a solution to (38) (MICHEL[12]). Or a direction of macroscopic strain rate is specified and the macroscopic stress of the evolution problem follows a path within P^{hom} which ends as t goes to $+\infty$ on the boundary of P^{hom}, at a point Σ_∞ which admits $\dot{E}^\circ$ as external normal to the extremal surface (SUQUET[17]).



\- Figure 9 -

For more details the reader is referred to DEBORDES & al where numerical computations show that the two above loadings give very similar results for P^{hom}.

<u>Influence of the boundary conditions on ∂V</u>.

The variety of boundary conditions which can be imposed on ∂V, leading to various possible choices of the space V_o, allows to define at least three different sets P^{hom} : $\hat{P}^{hom}$, P^{hom}_{per}, $\tilde{P}^{hom}$. In the definition of $\hat{P}^{hom}$ the stresses are supposed to be uniform on ∂V, in the definition of P^{hom}_{per} they are only supposed to be periodic, and no assumption on the stresses on ∂V are involved in the definition of $\tilde{P}^{hom}$. Since the other requirements contained in the definition (35) of P^{hom} are identical for the three sets, the following inclusions are easily stated

$$\hat{P}^{hom} \subset P^{hom}_{per} \subset \tilde{P}^{hom} \tag{43}$$

Using the embeddings $\tilde{V} \subset V_{per} \subset \hat{V}$, we derive the following inequalities on the plastic dissipations which also result from (43) :

$$\hat{D} \leqslant D_{per} \leqslant \tilde{D} \tag{44}$$

If periodic media are under consideration, (43) asserts that the assumption of uniform stresses on ∂V will give an underestimate of the strength, while the assumption of uniform strains on ∂V will overestimate the strength.

Comments :

P^{hom} has been introduced in the above form by the author [18], for periodic media. However previous works similarly based on limit analysis contained more or less explicitely the above definition (35) of P^{hom} : HILL [1], DRUCKER [19], SHU & al [20], M^c LAUGHLIN [21], LE NIZHERY [22] (+).

More recently DE BUHAN [23,24] reached a similar result for multi layered media which amounts to (35) for periodic stratifications and illustrated his work by interesting analytical determinations of P^{hom} in connection

(+) further references might be found in HASHIN [4] section 7 .

with Soil Mechanics problems. In a different direction, GURSON [25] has proposed a yield criterion for porous materials, and his arguments almost amount to the computation of $\mathcal{D}(\dot{E})$. He did not used the entire space $\mathcal{V}_o$, but rather a Riesz approximation of it, splitting the field $\bar{u}$ which enters (40) on a basis of displacement rates derived from solutions of linear problems.

4.3 COMPARISON EXPERIMENTS/NUMERICAL COMPUTATIONS

We go back to the experiments by LITEWKA & al reported in the introduction and compare them with numerical results taken from MICHEL [12] and MARIGO & al [11].

Rupture loads

In the tensile test reported on figure 1 the macroscopic stress tensor, when expressed in axis (1,2) , takes the form

$$\Sigma = \lambda \begin{pmatrix} \sin^2\alpha & \sin\alpha\ \cos\alpha & 0 \\ \sin\alpha\ \cos\alpha & \cos^2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = \lambda\Sigma^\circ(\alpha) \qquad (45)$$

We use the definition (38) of $\lambda_r(\alpha)$ where it appears as an upper bound :

$$\lambda_r(\alpha) = \sup\ \{\lambda \mid \lambda\Sigma^\circ(\alpha) \in P^{hom}\}$$

In order to solve (38) the computations are performed on the square perforation pattern, and the virgin material is idealized as an elastic-perfectly plastic one. Therefore the hardening part of the stress-strain curve is not correctly reproduced, but this lack of precision does not affect the value of the limit load. The elastic properties of the virgin material are specified in section 3 , and we note on figure 1 that its ultimate equivalent stress is

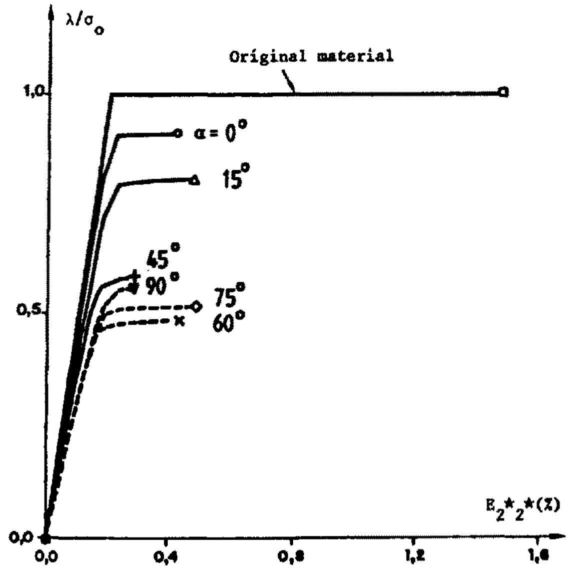
$$\sigma_o = 159 \text{ MPa} .$$

It will be supposed to obey the Von Mises criterion

$$\sigma_{eq} \leqslant \sigma_o .$$

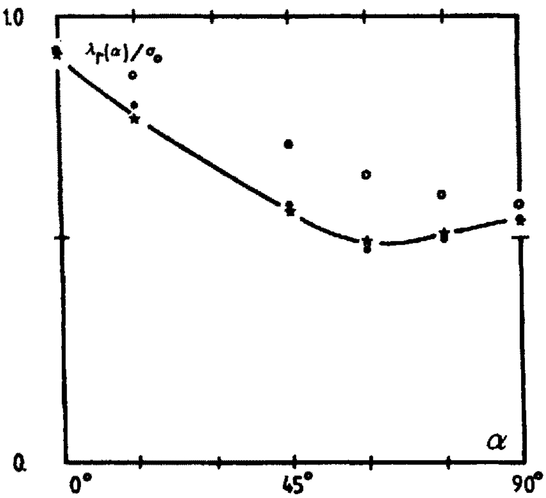
The specific numerical method used to solve (38) is described in details in [11] . Let us only comment briefly on the periodicity boundary conditions. In elastic problems on a r.v.e. exhibiting symmetries, we have reduced them to ordinary ones, mainly by means of the superposition principle. However, in the non-linear setting under consideration here, tensile stresses and shear stresses cannot be decoupled, and for the general stress $\Sigma^\circ(\alpha)$ we cannot get rid of the periodicity conditions. A survey of possible methods of resolution of problems involving periodic boundary conditions (penalty, elimination, Lagrangian...) is given in DEBORDES & al [10] .

We have plotted on figure 10 the external stress strain curves, computed on the idealized material at various inclinations α . We can deduce from this figure the values of the ultimate loads $\lambda_r(\alpha)$.



External stress / external strain (computed)

\- Figure 10 -



o Uniform strains on ∂V

● Periodicity conditions

★ Experiment [2]

Failure loads / orientation

\- Figure 11 -

Figure 11 reports the results of experiments, of the homogenization theory with periodicity conditions, and of the homogenization theory with uniform strain on ∂V . The agreement of the former theory with experiments is quite satisfactory, while the results of the latter (uniform strain) are overestimated in an obvious manner in agreement with previous considerations (43) .

Comments

This observation leads us to the following important comment.

In elasticity, the boundary conditions seemed to play a minor role since all local effects due to deviations in the boundary conditions were smoothed out. However, in plasticity local effects play an important role, and the deviations in boundary conditions are enlarged at the macroscopic scale. This fact has another interpretation. While it seems acceptable to model random distributions of elastic constituents by a periodic arrangement, such an idealization will be hazardous for elastic plastic constituents.

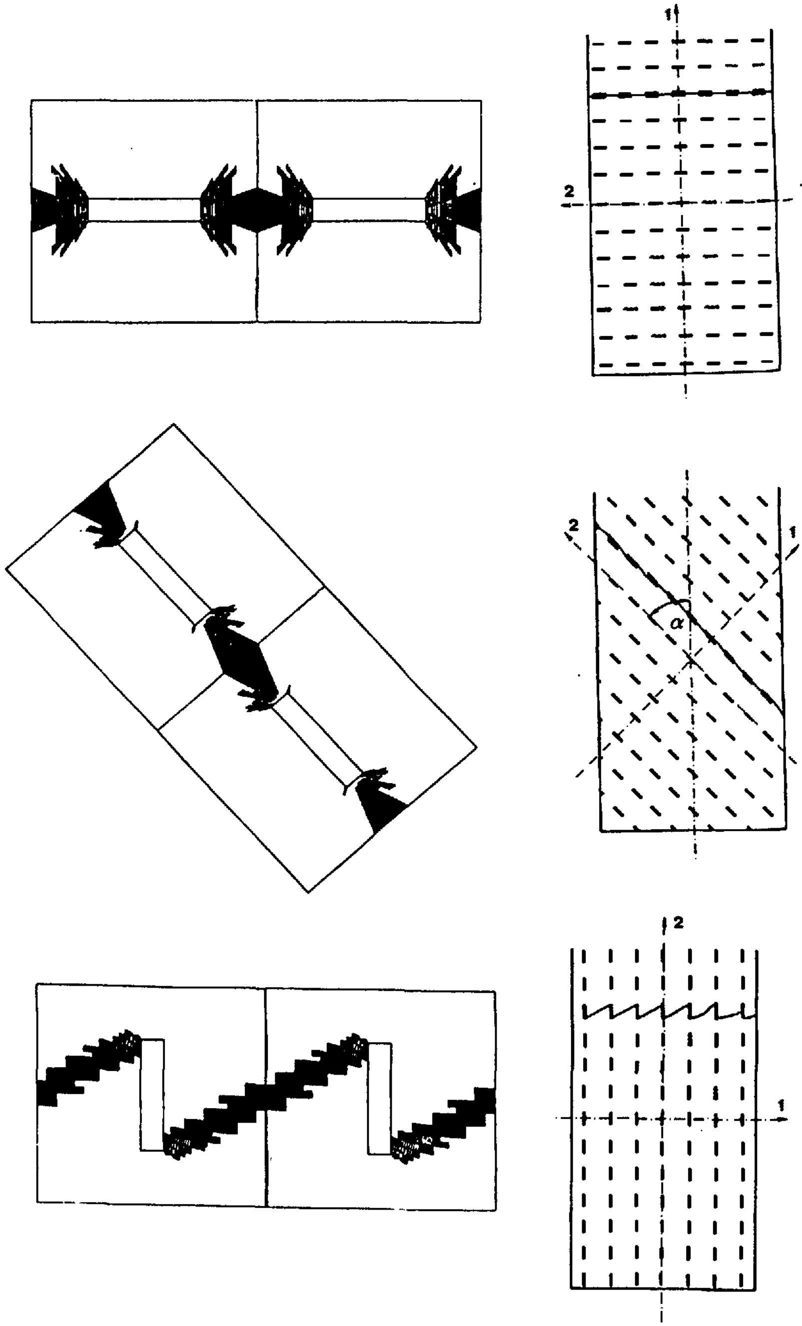
Rupture modes

Up to now the comparison experiments/computations was performed on macroscopic quantities (stiffnesses or rupture loads). This comparison can also been made at a more local level on the shape of the rupture modes. Figure 12 shows that the agreement is still good, but this holds true only for the periodic boundary conditions, since it is clear that when the plastic zone reaches the boundary of V the strain cannot be any more closed to a uniform strain on ∂V .

$\alpha = 90°$

$\alpha = 45°$

$\alpha \simeq 0°$



Computed

Experiment[2]

– Figure 12 : Failure Modes –

CHAPTER 5

5. ELASTIC PERFECTLY PLASTIC CONSTITUENTS

We now turn to the more difficult problem of describing the overall behavior of a material made of the assembly of elastic perfectly plastic constituents. In the stress-strain curve of such a material a hardening part is strongly expected. This hardening effect, due to a micro-stored elastic energy, will be described qualitatively and approximate models will be proposed. Most of the developments presented here are also valid for viscoplastic constituents.

The micro constitutive law reads as

$$\varepsilon(u) = \varepsilon^e + \varepsilon^p \quad , \quad \varepsilon^e = A : \sigma \tag{46}$$

and $\quad \sigma(y) \in P(y) \qquad$ for every y in V

$$\dot{\varepsilon}^p(y) : \bar{\sigma} - \sigma(y) \leq 0 \qquad \text{for every } \bar{\sigma} \text{ in } P(y)$$

5.1 MACROSCOPIC POTENTIALS

The major part of this paragraph follows the line of MANDEL's work[3] chap. 7 devoted to the macroscopic behavior of polycristalline aggregates.

Macroscopic plastic strain

We multiply (46) by the transposed tensor of elastic stress localization C^T, and we average on the r.v.e.

$$<C^T : \varepsilon> = <C^T : A : \sigma> + <C^T : \varepsilon^p> = <\sigma : A : C> + <C^T : \varepsilon^p>$$

A:C and σ are respectively an admissible strain field and an admissible stress field. By the equality of virtual power (7) we get

$$<C^T : \varepsilon> = <C^T> : <\varepsilon> = E \; ; \; <\sigma : A : C> = <\sigma> :< A : C> = A^{hom} : \Sigma$$

and

$$E = A^{hom} : \Sigma + <C^T : \varepsilon^p> \tag{47}$$

We recognize in $A^{hom} : \Sigma$ the elastic part of the macroscopic strain

$$E^e = A^{hom} : \Sigma = <C^T : A : \sigma> = <C^T : \varepsilon^e> ,$$

and therefore the plastic part of the macroscopic strain is given by

$$E^p = <C^T : \varepsilon^p> \tag{48}$$

It is worth noting that, generally speaking, neither the elastic part nor the plastic part of the macro strain is the average of its microscopic analogue (+) .

(+) a noticeable exception occurs when ε^p is compatible (i.e. derives from a displacement field). It can be the case for large strains, when elastic strains are small.

Macro internal energy

The averaging procedure for additive quantities described in section 2 allows us to compute the macroscopic internal energy as the average of the microscopic internal energy. Since the processes under consideration in this section are isothermal the micro internal evergy reduces to the elastic energy :

$$\bar{\rho}\,\mathcal{E} = \langle \rho \;\; e \rangle = \frac{1}{2} \langle (\varepsilon(u)-\varepsilon^P) : a : (\varepsilon(u) - \varepsilon^P) \rangle = \frac{1}{2} \langle \sigma : A : \sigma \rangle$$

We split the actual micro stress σ into two parts : the one which would occur if the material were perfectly elastic and a *self equilibrated residual stress tensor* (so called since it is the stress state under a null macro stress $\Sigma = 0$)

$$\sigma(y) = C(y) : \Sigma + \sigma^r(y) \tag{49}$$

Using this decomposition we get

$$\bar{\rho}\,\mathcal{E} = \frac{1}{2} \langle \sigma : A : \sigma \rangle = \frac{1}{2} \Sigma : \langle C^T : A : C \rangle : \Sigma + \langle \sigma^r : A : C \rangle \Sigma + \frac{1}{2} \langle \sigma^r : A : \sigma^r \rangle$$

But $A:C$ is an admissible strain field, while σ^r is a self equilibrated field. Applying (7) we see that the cross term $\langle \sigma^r : A : C \rangle$ vanishes. We are let with

$$\bar{\rho}\,\mathcal{E} = \frac{1}{2} \Sigma : A^{hom} : \Sigma + \frac{1}{2} \langle \sigma^r : A : \sigma^r \rangle \tag{50}$$

The first term in the above expression of $\mathcal{E}$ is the macroscopic elastic energy, and the second term is the *stored energy* : it is the elastic energy of the residual stresses and it is always positive except when the residual stresses vanish. It will be shown later on that this occurs only when the micro plastic strains are compatible, i.e. when they derive from a displacement field.

More about stored energy

In a recent work[26] CHRYSOCHOOS has reported microcalorimetric experiments performed on a AU4G in monotonic uniaxial tension. He has observed for this specific experiment that the stored energy reaches a threshold when the plastic strain increases. This means that the ratio stored energy/external work tends to 0 and not to 10 % as it is classically admitted by a somewhat hazardous interpretation of TAYLOR and QUINNEY's experiments. The limitation of the stored energy can receive an interpretation by means of the above arguments. Consider an assembly of elastic perfectly plastic constituents each of them obeying a Von Mises criterion (or any pressure insensitive criterion). It follows from the inequality (34) that the deviatoric part Σ^D of the macro stress tensor is limited. In a uniaxial tension test this deviatoric part can be expressed in terms of the only non vanishing component of Σ and this shows that Σ itself is bounded at any stage of the tension test. At the microscopic scale we know from the Von Mises criterion that σ^D is bounded. If we can prove that $\mathrm{Tr}\,\sigma$ is bounded at least in the space $L^2(V)$ then σ itself will be bounded in this space at any stage of the test. Therefore by (49)' the residual stresses σ^r will be bounded in $L^2(V)$ and their elastic energy, i.e. the energy stored along the tensile test will be limited. In order to prove that $p = -\frac{Tr\,\sigma}{3}$ is bounded in $L^2(V)$ we notice that the equilibrium equation yields

$$\frac{\partial p}{\partial x_i} = \frac{\partial \sigma^D_{ij}}{\partial x_j}$$

Since σ^D is bounded in $L^\infty(V)$, $\frac{\partial p}{\partial x_i}$ is bounded in $H^{-1}(V)$. A classical argument in the discussion of Navier-Stokes equations yields

$$|p - \langle p \rangle|_{L^2(V)} \leqslant C\left|\frac{\partial p}{\partial x_i}\right|_{H^{-1}(V)} \leqslant C'$$

But $\langle p \rangle = -\frac{Tr\,\Sigma}{3}$ is bounded by the above arguments. Therefore we have shown that p is bounded in $L^2(V)$. This completes the proof of the following result, evidenced by CHRYSOCHOOS's experiments : *in a monotonic uniaxial tension test, the stored energy of an assembly of elastic perfectly plastic materials is limited.* It is quite interesting to note, following the lines of CHRYSOCHOOS, that the classical models of kinematical or isotropic hardening do not ensure this limitation of the stored energy, although the mechanical behavior (i.e. the stress strain curve) predicted can fit experiments in a satisfactory manner for simple loadings. If we remember the role played by the stored energy in shakedown or accommodation analysis it becomes obvious that the above property is not an academic one if it can be generalized to more complex loadings. It also evidences the role that thermal experiments should play in the determination of the mechanical behavior of an aggregate.

Plastic work

The averaging procedure for additive functions allows us again to compute the macroscopic dissipation as the average of the microscopic one :

$$\mathcal{D} = <d> = <\sigma : \dot{\varepsilon}^p> = <C\,\Sigma : \dot{\varepsilon}^p> + <\sigma^r : \dot{\varepsilon}^p>$$

$$= \Sigma : <C^T \dot{\varepsilon}^p> + <\sigma^r : \dot{\varepsilon}^p> = \Sigma : \dot{E}^p + <\sigma^r : \dot{\varepsilon}^p>$$

In order to carry on the computation of $\mathcal{D}$ we note that the field of residual stresses σ^r has the following properties, which result from its definition

$$\left.\begin{array}{l} \sigma^r \in SE \\ A\sigma^r + \varepsilon^p = \varepsilon(u^r) \end{array}\right\} \qquad (51)$$

where $\varepsilon(u^r) = \varepsilon(u) - A : C : \Sigma$ is an admissible strain field. Therefore the macroscopic dissipation amounts now to :

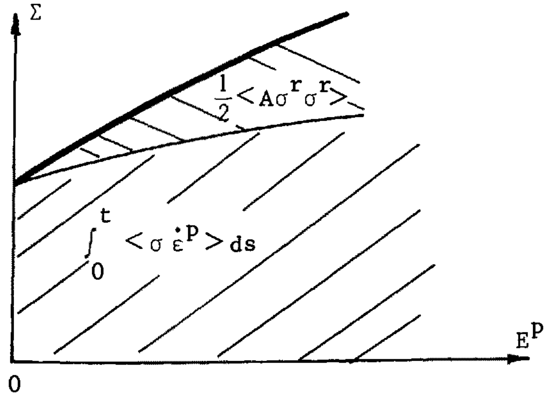
$$\mathcal{D} = \Sigma : \dot{E}^p + <\sigma^r : \varepsilon(\dot{u}^r)> - <\sigma^r : A : \dot{\sigma}^r> \qquad (52)$$

Since σ^r is self equilibrated the second term in (52) vanishes

$$\mathcal{D} = \Sigma : \dot{E}^p - <\sigma^r : A : \dot{\sigma}^r> = <\sigma : \dot{\varepsilon}^p>$$

Therefore the macro plastic work-rate $\Sigma : \dot{E}^p$ does not reduce to the average of the micro one $<\sigma : \dot{\varepsilon}^p>$, and the difference between these quantities is the elastic energy-rate $<\sigma^r : A : \dot{\sigma}^r>$ due to the development of residual stresses. At the microscopic level the plastic work-rate is entirely dissipated, while at the macroscopic level it is partly dissipated (in the plastic micro mechanisms) and partly stored in the increase of the elastic energy of residual stresses :

$$\Sigma : \dot{E}^p = \langle \sigma : \dot{\varepsilon}^p \rangle + \langle \sigma^r : A : \dot{\sigma}^r \rangle \ .$$



– Figure 13 –

Stability in Drucker's sense

Since the constituents are assumed to be elastic perfectly plastic we know that at every microscopic point the following equality holds true

$$\dot{\sigma} : \dot{\varepsilon}^p = 0 \ .$$

The decomposition (49) of σ yields

$$\langle C\, \dot{\Sigma} : \dot{\varepsilon}^p \rangle + \langle \dot{\sigma}^r : \dot{\varepsilon}^p \rangle = \langle \dot{\sigma} : \dot{\varepsilon}^p \rangle = 0$$

But it follows from (51) and (48) that

$$<\dot{\sigma}^r : \dot{\varepsilon}^p> = -<\dot{\sigma}^r : A : \dot{\sigma}^r> \ , \ <C : \dot{\Sigma} : \dot{\varepsilon}^p> = \dot{\Sigma} : <C^T : \dot{\varepsilon}^p> = \dot{\Sigma} : \dot{E}^p$$

Therefore

$$\dot{\Sigma} : \dot{E}^p = <\dot{\sigma} : \dot{\varepsilon}^p> + <\dot{\sigma}^r : A : \dot{\sigma}^r> \geqslant 0 \tag{53}$$

This last inequality shows that the composite material is stable in Drucker's sense at the macroscopic level. It should be noted from (53) that, since $<\dot{\sigma}^r : A : \dot{\sigma}^r>$ is always non negative, we have

$$\dot{\Sigma} : \dot{E}^p \geqslant <\dot{\sigma} : \dot{\varepsilon}^p> \quad .$$

We could express this inequality by saying in a somewhat loosy manner that the change of scale stabilizes the material.

<u>Macroscopic yield surface</u>

We now assume that the composite material has been loaded up to microscopic stress state $\sigma(y)$ with residual stresses $\sigma^r(y)$. The macroscopic yield locus is the set of macroscopic stresses $\Sigma^\star$ which can be reached from the present state Σ by an elastic path, along which the residual stresses remain unchanged. The microscopic state $\sigma^\star$ satisfies

$$\sigma^\star(y) - \sigma(y) = C(y) : (\Sigma^\star - \Sigma) \tag{54}$$

and $\sigma^\star(y) = C(y) : \Sigma^\star + \sigma^r(y)$.

We notice that the condition

$$\sigma^{\star}(y) \in P(y) \quad \text{for every} \quad y \quad \text{in} \quad V$$

is equivalent to

$$\Sigma^{\star} \in C(y)^{-1} : (P(y) - \{\sigma^r(y)\}) \quad \text{for every} \quad y \quad \text{in} \quad V \quad \text{and therefore}$$

$$\Sigma^{\star} \in P^{hom}(\{\sigma^r\}) = \bigcap_{y \in V} C(y)^{-1} [P(y) - \{\sigma^r(y)\}] \tag{55}$$

The macro yield locus $P^{hom}(\{\sigma^r\})$ is a convex set (intersection of convex sets). Its determination at a given time t requires the knowledge of the whole set of residual stresses. Therefore it is not possible to entirely eliminate the microscopic level from the macroscopic behavior, as it is the case in the elastic setting. However we can analyse in a qualitative manner the way in which the macroscopic yield locus is obtained in the stress space :

★ The set $P(y) - \{\sigma^r(y)\}$ is translated from the original set $P(y)$, without change in shape, or size. This operation results in a kinematic hardening.

★ Multiplying the previous set by $C(y)^{-1}$ amounts to a rotation and an anisotropic expansion of this set. This operation does not reduce to isotropic hardening although it bears some resemblance with it.

★ The last operation is the intersection over all y in V. This is a complex operation including a change in shape, a change in size, and a change in the center of the convex set. If the intersection is to be taken over a finite set of points $y^{(+)}$ the boundary of the set $P^{hom}(\{\sigma^r\})$ will probably exhibit vertices. Such vertices will be smoothed off if the intersection is taken over an infinite set of points $y^{(++)}$. It should be noted that this smoothing effect is due to the non uniformity of the residual stresses and therefore take its origin mainly in the heterogeneous *elasticity* of the composite.

5.2 STRUCTURE OF THE MACROSCOPIC CONSTITUTIVE LAW

We now try to analyse in a qualitative way the structure of the macroscopic constitutive law. We claim that the state variables are :

★ the macro strain E

★ the *whole field* of micro plastic strains $\{\varepsilon^P(y), y \in V\}$. This means an infinite number of internal variables.

Indeed, once these variables are specified the actual micro stress state can be derived as follows :

a) Σ is deduced from E and $\{\varepsilon^P\}$ by (47)

b) σ^r can be computed as the solution of the elastic problem (51) where ε^P is considered as a known quantity (analogous to a thermal strain). The field σ^r is a linear functional of the field ε^P

(+) this is the case if yielding is likely to occur on a finite set of planes (monocristal) or if the residual stresses $\sigma^r(y)$, and the yield loci $P(y)$ are piecewise constant.

(++) this is the case if the r.v.e. is a polycristal with a large number of grains.

$$\sigma^r = - R : \varepsilon^P \quad \text{i.e.} \quad \sigma^r(y) = - \int_V R(y,y')\varepsilon^P(y')\,dy \tag{56}$$

The integro-differential operator R can be expressed easily in terms of the Green function of the elastic problem (51) , but we shall not need its exact expression.

Once the state variables are identified one has to compute the internal energy of the material. We already know by a previous computation that

$$\bar{\rho}\,\mathcal{E} = \frac{1}{2}\,\Sigma : A^{hom} : \Sigma + \frac{1}{2} < \sigma^r : A : \sigma^r >$$

and we now express $\mathcal{E}$ in terms of the state variables $(E,\{\varepsilon^P\})$

$$\bar{\rho}\,\mathcal{E} = \frac{1}{2}\,(E-E^P) : a^{hom} : (E-E^P) + \frac{1}{2} < \sigma^r : (\varepsilon(u^r) - \varepsilon^P) >$$

Taking into account (56) and the fact that σ^r is self equilibrated we get

$$\bar{\rho}\,\mathcal{E}(E,\{\varepsilon^P\}) = \frac{1}{2}\,(E-E^P) : a^{hom}(E-E^P) + \frac{1}{2} < R\varepsilon^P : \varepsilon^P > \tag{57}$$

The *state laws* relate the thermodynamical forces associated with the state variables, and the state variables themselves. The thermodynamical forces are defined as

$$\bar{\rho}\,\frac{\partial\mathcal{E}}{\partial E} \quad \text{and} \quad -\bar{\rho}\,\frac{\partial\mathcal{E}}{\partial\{\varepsilon^P\}} \quad ,$$

and their computation is immediate

$$\bar{\rho}\,\frac{\partial\mathcal{E}}{\partial E} = a^{hom}(E-E^P) = \Sigma$$

and for a virtual field of microscopic plastic strains $\delta\,\varepsilon^P$

$$<-\,\bar{\rho}\,\frac{\partial\,\mathcal{E}}{\partial\{\varepsilon^P\}} : \delta\,\varepsilon^P> \;=\; <a^{hom}(E-E^P)\frac{\partial E^P}{\partial\{\varepsilon^P\}} : \delta\,\varepsilon^P> \;-\; <R\,\varepsilon^P : \delta\,\varepsilon^P>$$

$$= <\Sigma : C^T : \delta\,\varepsilon^P> - <R\varepsilon^P : \delta\,\varepsilon^P> = <\delta\,\varepsilon^P : (C:\Sigma+\sigma^r)> = <\delta\,\varepsilon^P : \sigma>$$

The thermodynamical force associated with the state variable $\{\varepsilon^P\}$ is the micro stress field $\{\sigma\}$. Setting

$$\alpha = \{\varepsilon^P\}\ ,\quad A = \{\sigma\}\ ,\quad \mathcal{P} = \{\tau \in \varepsilon(\mathcal{V}_o)^{\perp}\ ,\tau(y) \in P(y) \ \text{ for every } y \text{ in } V\}$$

we can recognize the generalized standard form [27,28] of the macro constitutive law :

$$\left.\begin{array}{lll} \text{state laws} & \Sigma = \bar{\rho}\,\dfrac{\partial\mathcal{E}}{\partial E}\ , & A = -\,\bar{\rho}\,\dfrac{\partial\mathcal{E}}{\partial\alpha} \\[2ex] & A \in \mathcal{P} & \\[2ex] \text{complementary laws} & & \\ & (\dot{\alpha}\ ,\ A'-A) \leqslant 0 & \forall\, A' \in \mathcal{P} \end{array}\right\} \quad (58)$$

However, this information on the structure of the macroscopic law is of little pratical importance since the constitutive law involves a infinite number of internal variables α . The next section will be devoted to the description of more useful, though approximate, models.

Remark. If the constituents are viscoelastic or viscoplastic a result similar to (58) holds true. Indeed the micro constitutive law now reads as

$$\varepsilon(\dot{u}) = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{an} = A(y)\dot{\sigma} + \frac{\partial\varphi}{\partial\sigma}(y,\sigma) \tag{59}$$

where φ is the potential which defines the anelastic part of the strain rate. The relations (47) and (48) still define the elastic and anelastic parts of the macroscopic strain. Moreover following RICE [29] it can be shown that the composite admits a *macroscopic potential* from which the anelastic part of the strain rate can be derived :

$$\dot{E}^{an} = \frac{\partial\Phi}{\partial\Sigma}(\Sigma,\sigma^{r}) \tag{60}$$

where $\Phi(\Sigma,\sigma^{r}) = \langle\varphi(y,\sigma)\rangle = \langle\varphi(y, C:\Sigma+\sigma^{r}\rangle$.

The complete form of the macroscopic constitutive law is

$$\dot{E} = A^{hom}\,\dot{\Sigma} + \frac{\partial\Phi}{\partial\Sigma}(\Sigma,\sigma^{r}) \tag{61}$$

where the residual stresses σ^{r} are found as the solution of the microscopic problem

$$\left.\begin{array}{l} \varepsilon(\dot{u}^{r}) = A\,\dot{\sigma}^{r} + \dfrac{\partial\Phi}{\partial\sigma^{r}}(\Sigma,\sigma^{r}) \\ \sigma^{r} \text{ self equilibrated, } \varepsilon(u^{r}) \text{ admissible strain field.} \end{array}\right\}$$

Once more the macroscopic and the microscopic levels are coupled by the presence of residual stresses.

5.3 APPROXIMATE MODELS

Once the complexity of the homogenized law is recognized we turn to approximate models in order to obtain more quantitative results. These approximate models are based on an a priori feeling of the microscopic distribution of plastic strains, or of residual stresses and more generally on the way in which yielding occurs at the microscopic level.

Piecewise constant plastic strains

In this first approximate model we replace the constraints

$$\sigma(y) \in P(y) \quad \textit{for every} \quad y \quad \textit{in} \quad V \tag{62}$$

by the following weaker requirements

$$\Sigma^{(i)} \in P_i \qquad i = 1, \ldots, n \tag{63}$$

where V has been partitionned into $V_1, \ldots, V_n$, where Σ_i is the partial average of the microscopic stress on the i^{th} phase V_i

$$\Sigma^{(i)} = \frac{1}{|V_i|} \int_{V_i} \sigma(y)\,dy = \langle \sigma \rangle_i \quad ,$$

and where P_i is the typical yield locus of V_i. We note that

$$\Sigma = c_i\, \Sigma^{(i)} \qquad \text{with} \qquad c_i = \frac{|V_i|}{|V|} \quad .$$

The set P reduces to

$$P = \{\sigma \in \varepsilon(V)^{\perp} \;\; ; \;\; \Sigma^{(i)} = <\sigma>_i \in P_i \;\; , \;\; i = 1,\ldots,n\}$$

Then the normality law, expressed at the microscopic level and averaged over each phases V_i , reads as

$$\sigma \in P$$

$$<\dot{\varepsilon}^P(y) : \bar{\sigma}(y) - \sigma(y)>_i \leqslant 0 \quad \text{for every } \bar{\sigma} \text{ in } P \text{ , } i=1,\ldots,n \tag{64}$$

Taking $\bar{\sigma} = \sigma + \sigma^{\star}$ where $\sigma^{\star}$ is chosen such that

$$<\sigma^{\star}>_i = 0 \quad , \quad \sigma^{\star} \in \varepsilon(V_o)^{\perp} \tag{65}$$

We see that for every $i = 1,\ldots,n$, and for every $\sigma^{\star}$ satisfying (65)

$$<\dot{\varepsilon}^P : \sigma^{\star}>_i = 0 \tag{66}$$

The classical theory of Lagrange's multipliers shows that $\dot{\varepsilon}^P$ must be constant on each phase V_i

$$\dot{\varepsilon}^P(y) = \dot{E}^P_i \qquad \text{for all } y \text{ in } V_i \text{ .}$$

After a time integration we deduce that ε^P must be constant on V_i . The internal variables $(E,\{\varepsilon^P(y)\})$ reduce now to the finite set $(E,E^P_1,\ldots,E^P_n)$. For the sake of simplicity we shall assume that n equals 2 , i.e. that the plastic strain depends on only two independent variables E^P_f and E^P_m

$$\varepsilon^p(y) = E^p_m \, \theta_m(y) + E^p_f \, \theta_f(y) \quad , \tag{67}$$

where $\theta_m(y) = 1$ in the phase V_m and 0 in $V_f = V - V_m$ (similar definition for θ_f)[(+)]. The microscopic constitutive law and the localization problem amount to find u and σ such that

$$\left.\begin{array}{l} \varepsilon(u) = E + \varepsilon(u^\star) = A : \sigma + E^p_m \, \theta_m + E^p_f \, \theta_f \\ \\ \mathrm{div}\, \sigma = 0 \quad \text{and boundary conditions} \end{array}\right\} \tag{68}$$

This problem bears a strong resemblance with (11) and, since it is linear with respect to E , E^p_m , E^p_f its solution $u^\star$ can be split into

$$u^\star = E\,\chi + E^p_m \, \chi^p_m + E^p_f \, \chi^p_f \quad ,$$

where χ has been defined in section 3 , and where χ^p_m and χ^p_f are solutions of

$$\left.\begin{array}{l} \chi^p_m \in \mathcal{V}_o \quad \text{and for every} \quad v \quad \text{in} \quad \mathcal{V}_o \\ \\ <\varepsilon(v) : a : \varepsilon(\chi^p_m)> \; = - <\varepsilon(v) : a : \theta_m> \end{array}\right\} \tag{69}$$

(similar definition for χ^p_f) .

The microscopic strain reads as

(+) the use of the subscripts f and m indicates that we have in mind a matrix/fiber composite.

$$\varepsilon(u) = D : E + E^P_m\ \varepsilon(\chi^P_m) + E^P_f\ \varepsilon(\chi^P_f) \qquad (70)$$

and this expression allows us to compute the macroscopic energy by the averaging procedure (2). Computing the thermodynamical forces associated with E, E^P_m, E^P_f yields

$$\bar{\rho}\ \frac{\partial \mathcal{E}}{\partial E} = <\rho\ \frac{\partial e}{\partial E}> = <(\varepsilon(\frac{\partial u}{\partial E}) - \frac{\partial \varepsilon^P}{\partial E}) : a : (\varepsilon(u) - \varepsilon^P)>$$

$$= <D^T : a : (\varepsilon(u) - \varepsilon^P)> = <D^T : \sigma> = \Sigma \quad .$$

In the same way a straightforward computation shows that :

$$-\bar{\rho}\ \frac{\partial \mathcal{E}}{\partial E^P_m} = c_m \langle\sigma\rangle_m = c_m\ \Sigma^{(m)}\ ,\ -\bar{\rho}\ \frac{\partial \mathcal{E}}{\partial E^P_f} = c_f \langle\sigma\rangle_f = c_f\ \Sigma^{(f)}$$

Moreover we derive from the averaged normality law (64) the following inequalities :

$$\dot{E}^P_m : \Sigma^{\star} - \Sigma^{(m)} = <\dot{\varepsilon}^P : \Sigma^{\star} - \sigma>_m \leqslant 0 \quad \text{for every} \quad \Sigma^{\star} \in P_m$$

$$\dot{E}^P_f : \Sigma^{\star\star} - \Sigma^{(f)} = <\dot{\varepsilon}^P : \Sigma^{\star\star} - \sigma>_f \leqslant 0 \quad \text{for every} \quad \Sigma^{\star\star} \in P_f$$

Now we can give the general form of the macroscopic law : after resolution of the problems (13) (69) we can compute the macroscopic internal energy $\bar{\rho}\,\mathcal{E}$ by the averaging process (2). Then we set

$$\alpha = (E^P_m, E^P_f)\ ,\ A = (c_m\ \Sigma^{(m)}\ ,\ c_f\ \Sigma^{(f)})\ ,\ P = (c_m\ P_m \times c_f\ P_f)$$

the macroscopic law has the standard form :

$$\Sigma = \bar{\rho}\ \frac{\partial \mathcal{E}}{\partial E} \quad , \quad A = -\bar{\rho}\ \frac{\partial \mathcal{E}}{\partial \alpha}$$

$A \in P$ and for every $A^\star$ in P

$(\dot{\alpha} : A^\star - A) \leqslant 0$

Remark. The assumption that θ_m and θ_f are piecewise constant is not essential : therefore we can give to them more general values modelling a non uniform distribution of plastic strains at the microscopic level. The model is still valid provided that the equality (67) holds true.

Averaged criterion (MICHEL[12])

In the second approximate model we assume that the yield criterion is satisfied *in average*. Specifically it is assumed that the constraints (62) , which take the form

$f(y,\sigma(y)) \leqslant 0$ for every y in V

are replaced by one inequality :

$$<f(y,\sigma)> \leqslant 0 \tag{71}$$

and the set P becomes

$$P = \{\sigma \in \varepsilon(V_o)^\perp \quad ; \quad <f(y,\sigma)> \leqslant 0\} \quad .$$

The normality law for P reads as

$$\left.\begin{array}{l} \sigma \in P \\ \\ \dot{\varepsilon}^P(y) = \dot{\lambda} \dfrac{\partial f}{\partial \sigma}(y,\sigma(y)) \end{array}\right. \tag{72}$$

and there is only *one* plastic multiplier $\dot{\lambda}$ for the whole r.v.e. V since the yield condition is expressed by a single inequality (71) . As pointed out by MICHEL [12] the problem (51) of computing the residual stresses takes a very simple form if we assume that the yielding is governed by the microscopic elastic energy

$$f(y,\sigma) = \frac{1}{2}\,\sigma : A : \sigma - k \,. \tag{73}$$

In this eventuality (72) gives

$$\dot{\varepsilon}^p = \dot{\lambda}\, A : \sigma$$

and the problem (51) of evaluating the residual stresses is equivalent to

$$\left.\begin{array}{l} \sigma^r \in SE \\ A : \dot{\sigma}^r + \dot{\lambda}\, A : \sigma^r = \varepsilon(\dot{u}^r) - \dot{\lambda}\, A : C : \Sigma \end{array}\right\} \tag{74}$$

If we remember that $A : C : \Sigma$ is an admissible strain field we obtain that σ^r is solution of the following evolution problem

$$\left.\begin{array}{l} \sigma^r \in SE \\ <\tau : A : \dot{\sigma}^r> + \dot{\lambda}<\tau : A : \sigma^r> = 0 \quad \text{for every} \quad \tau \quad \text{in} \quad SE \end{array}\right\} \tag{75}$$

the solution of which is

$$\sigma^r(t) = e^{-(\lambda(t)-\lambda_o)}\sigma^r(0) \ . \tag{76}$$

The remarkable fact in (76) is that we are able to compute the whole field of residual stresses as a function of a single parameter ξ :

$$\xi(t) = e^{-(\lambda(t)-\lambda_o)} \tag{77}$$

We note that the yield condition (71) becomes

$$\frac{1}{2} < (C : \Sigma + \sigma^r) : A : (C : \Sigma + \sigma^r) > - < k > \leqslant 0$$

$$\frac{1}{2}\Sigma : A^{hom} : \Sigma + \frac{1}{2} h\ \xi^2 - < k > \leqslant 0 \tag{78}$$

where $h = < \sigma^r(0) : A : \sigma^r(0) >$.

This last inequality (78) shows that the composite undergoes *isotropic hardening*, where the hardening parameter is ξ . It turns out that the state variables of this model are (E, E^p, ξ) , and the following lines will show that this choice of state variables lead to a generalized standard form of the macroscopic constitutive law. Indeed the general expression of the internal energy reduces here to

$$\overline{\rho}\ (E, E^p, \xi) = \frac{1}{2}(E - E^p) : a^{hom} : (E - E^p) + \frac{1}{2} h\ \xi^2$$

since $< \sigma^r : A : \sigma^r > = h\ \xi^2$.

The thermodynamical forces associated with E , E^p and ξ are

$$\overline{\rho}\ \frac{\partial \mathcal{E}}{\partial E} = \Sigma\ , \quad - \overline{\rho}\ \frac{\partial \mathcal{E}}{\partial E^P} = \Sigma\ , \quad - \rho \frac{\partial \mathcal{E}}{\partial \xi} = - h\ \xi\ = A^{\xi} \quad . \tag{79}$$

We set

$$F(\Sigma, A^{\xi}) = \frac{1}{2}\ \Sigma : A^{hom} : \Sigma + \frac{1}{2}\ \frac{(A^{\xi})^2}{h} - <k> \quad ,$$

and we note that according to (78)

$$F(\Sigma, A^{\xi}) \leqslant 0 \tag{80}$$

$$\dot{E}^P = <C^T : \dot{\varepsilon}^P> = <C^T : \dot{\lambda}\ A : \sigma> =$$

$$= \dot{\lambda}\ <C^T : A : C> : \Sigma + \dot{\lambda}\ <C^T : A : \sigma^r>$$

But $<C^T : A : \sigma^r> = <\sigma^r : A : C> = 0$ by (7) .

We are let with

$$\dot{E}^P = \dot{\lambda}\ A^{hom} : \Sigma = \dot{\lambda}\ \frac{\partial F}{\partial \Sigma}\ (\Sigma, A_{\xi}) \tag{81}$$

On the other hand

$$\dot{\xi} = - \dot{\lambda}\ \xi\ = \dot{\lambda}\ \frac{\partial F}{\partial A_{\xi}}(\Sigma, A_{\xi}) \tag{82}$$

It can be checked that $\dot{\lambda}$ obeys the usual requirements of a plastic multiplier. The macroscopic constitutive law which consists in (79)-(82) is therefore a generalized standard law. We can give further interpretations of the parameters entering the model :

. the hardening modulus h is the elastic energy of the *initial residual stresses*.

. the size of the loading surface defined by F (or similarly by (78)) is

$$\langle k \rangle - \frac{1}{2} h \, \xi^2$$

But since $\dot{\lambda}$ is positive, λ increases and ξ decreases by (77) .

Therefore the size of the macroscopic loading surface increases and approaches k asymptotically :

. the plastic multiplier is proportional to the macroscopic dissipation. Indeed

$$D = \langle d \rangle = \langle \sigma : \dot{\varepsilon}^p \rangle = \langle \sigma : \dot{\lambda} : A : \sigma \rangle$$

$$= \dot{\lambda} \, \langle \sigma : A : \sigma \rangle$$

This last term either vanishes if $\langle \sigma : A : \sigma \rangle - \langle k \rangle < 0$ or is equal to $\dot{\lambda} \, k$ if $\langle \sigma : A : \sigma \rangle - \langle k \rangle = 0$. Thus

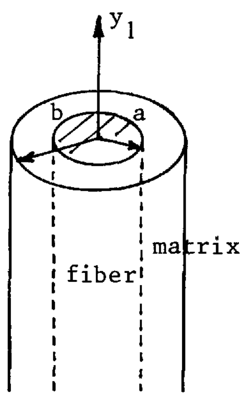
$$D = \dot{\lambda} \langle k \rangle$$

. The stored energy *decreases* along any loading path, since ξ decreases.

DVORAK & RAO's model for unidirectional fiber composites

A significant advance in the modelling of the plastic behavior of unidirectional fiber composite has been achieved by DVORAK & RAO[30] who proposed a model of kinematic hardening briefly described here below.

The matrix is supposed to be elastic perfectly plastic and to obey the normality rule, with yield function f. The fibers are elastic and aligned in the direction y_1. The elementary volume is a composite cylinder of external radius b, in which a fiber of radius a is embedded in the matrix. The loadings under consideration are axisymmetric : traction or compression along y_1, and equiaxial stresses in the plane (y_2,y_3) and this loading is represented by a two components vector $\underline{\Sigma}$



- Figure 14 -

$$\underline{\Sigma} = (\Sigma_{11}, \frac{1}{2}(\Sigma_{22} + \Sigma_{33}))$$

We shall denote by $\underline{C}$ the part of the localization tensor which yields the microscopic state in terms of $\underline{\Sigma}$ in the elastic range

$$\sigma(r) = \underline{C}(r)\underline{\Sigma}$$

where $r = (y_2^2 + y_3^2)^{1/2}$. In the elastic regime the maximal local stresses are located in the fiber and at the fiber-matrix interface. The first assumption of DVORAK & RAO's model is that this property still holds true in the elastic plastic range whatever is the field of (axisymmetric) initial residual stresses. More specifically numerical experiments performed by these authors show that it is reasonable to assume that *under any axisymmetric complex loading yielding occurs first at the fiber/matrix interface.* (83)

Assume that the composite is loaded from an elastic state in which the residual stresses are σ^r . Yielding at the macroscopic level occurs as soon as plasticity occurs at the microscopic level, i.e. by virtue of the above assumption as soon as the stresses at the interface reach the yield limit. Therefore the macroscopic yielding starts as soon as

$$f(\underset{\sim}{C}(a)\underset{\sim}{\Sigma} + \sigma^r(a)) = 0 \qquad (84)$$

Let us set $\underset{\sim}{X} = - \underset{\sim}{C}(a)^{-1}\ \sigma^r(a)$. Then the condition (84) of macroscopic yielding reads as :

$$g(\underset{\sim}{\Sigma} - \underset{\sim}{X}) = 0 \ .$$

The above assumption, indeed rather weak, has allowed to derive the following remarkable result : the macroscopic yield surface undergoes *kinematic hardening*. Its center $\underset{\sim}{X}$ moves in the space of axisymmetric loadings, while its shape (characterized by g) does not change. It is readily seen that this result is a general one, i.e. that if yielding at the microscopic level turns out to occur first in the same points, then the macroscopic yield surface undergoes kinematic hardening.

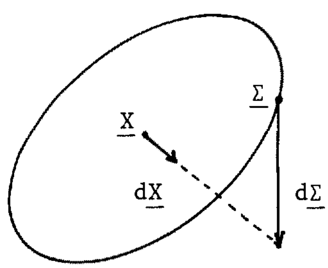
Hardening rule

The generality of the assumption (83) does not permit to compute the microscopic fields of plastic strains or of residual stresses as in the two previous examples. Therefore we cannot rely on thermodynamical arguments to derive the hardening rule and the flow rule. However this can be done by direct means. The loading condition expresses the orthogonality between $d\underline{\Sigma} - d\underline{X}$ and grad g .

$$0 = dg = \frac{\partial g}{\partial \underline{\Sigma}_1}\,(d\underline{\Sigma}_1 - d\underline{X}_1) + \frac{\partial g}{\partial \underline{\Sigma}_2}\,(d\underline{\Sigma}_2 - d\underline{X}_2) = 0$$

Therefore there exists a multiplier $d\mu$ such that

$$d\underline{X} = d\underline{\Sigma} - d\mu \begin{pmatrix} -\dfrac{\partial g}{\partial \underline{\Sigma}_2} \\ \\ \dfrac{\partial g}{\partial \underline{\Sigma}_1} \end{pmatrix} \tag{85}$$



- Figure 15 -

In order to determine the multiplier $d\mu$ we assume that the increment $d\underline{X}$ of the internal stress $\underline{X}$, points from $\underline{X}$ in the direction of the new stress state $\underline{\Sigma} + d\underline{\Sigma}$ (see figure). The vectors $d\underline{X}$ and $\underline{\Sigma} - \underline{X} + d\underline{\Sigma}$ must be colinear. Neglecting the second order terms we obtain

$$d\mu = \frac{(\underline{\Sigma}_2 - \underline{X}_2)\,d\underline{\Sigma}_1 - (\underline{\Sigma}_1 - \underline{X}_1)\,d\underline{\Sigma}_2}{(\underline{\Sigma}_1 - \underline{X}_1)\dfrac{\partial g}{\partial \underline{\Sigma}_1} + (\underline{\Sigma}_2 - \underline{X}_2)\dfrac{\partial g}{\partial \underline{\Sigma}_2}} \tag{86}$$

The hardening law is completely determined by (85) and (86).

Flow rule

The plastic part of the macroscopic strain satisfies

$$\begin{pmatrix} d\underline{E}_1^p \\ \\ d\underline{E}_2^p \end{pmatrix} = d\lambda \begin{pmatrix} \dfrac{\partial g}{\partial \underline{\Sigma}_1} \\ \\ \dfrac{\partial g}{\partial \underline{\Sigma}_2} \end{pmatrix}$$

DVORAK & RAO determine the macroscopic plastic multiplier $d\lambda$ by assuming further properties of the localization of stresses during the loading process. Their assumptions, based on numerical calculations, yield the following expression for $d\lambda$:

$$d\lambda = \frac{1}{\dfrac{\partial g}{\partial \underline{\Sigma}_1}} \left(\frac{1}{c_f E_f} - \underline{A}_{12}^{hom}\right) d\underline{X}_1 + \left(\frac{-2\nu_f + (1-c_f)/c_f}{E_f}\right) d\underline{X}_2$$

where c_f denotes the fiber volume fraction and $\underline{A}_{12}^{hom}$ and $\underline{A}_{22}^{hom}$ the compliances of the composite relating $\underline{\Sigma}$ and $\underline{E}$.

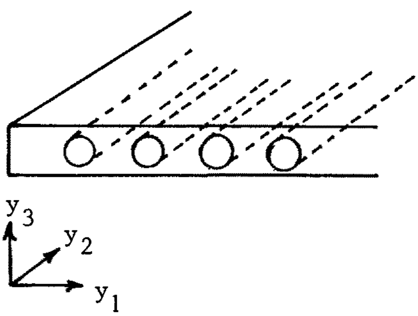
CONCLUSIONS

The behavior of composites in the nonlinear and inelastic range is still a widely open subject where all further contributions would be greatly appreciated. We have tried to show that failure of ductile heterogeneous materials could be predicted in a satisfactory manner by means of a limit analysis study. Moreover we have proposed a few simplified models describing the macroscopic hardening of a composite. These models are based on crude approximation on the microscopic fields, or restrict the attention to specific loadings, and there is an important need of further work in this direction.

PROBLEMS

Part 2 2.1 Show that there exists infinitely many boundary conditions ensuring the validity of the equality of virtual work and leading to a well posed problem in (13) . Consider for instance an arbitrary partition of V into $\partial_1 V$ and $\partial_2 V$ and impose uniform stresses on $\partial_1 V$ and uniform strains on $\partial_2 V$.

2.2 Consider a thin sheet periodically perforated in its thickness. Show that the natural boundary conditions on ∂V are



- Figure 16 -

$\sigma_{i3} = 0 \quad 1 \leq i \leq 3 \quad$ for $\quad y_3 = \pm h/2$

$u_\alpha = E_{\alpha\beta} y_\beta + u_\alpha^\star \quad u_\alpha^\star$ periodic on ∂V

$\sigma_{\alpha\beta} n_\beta$ antiperiodic on ∂V $(1 \leq \alpha, \beta \leq 2)$.

Prove that these boundary conditions ensure the validity of (7) .

2.3 In the periodic setting can you relate (6) with the asymptotic expansion theory.

Part 3 3.1 Prove that $\varepsilon(V_o)$ is a Hilbert space, for the three possibilities $V_o = \hat{V}$, V_{per}, $\tilde{V}$, when endowed with the scalar product $<\varepsilon : \varepsilon'>$. In a first step prove that

$$(\exists\, c > 0)(\forall u \in H^1(V)^3) \ |u - <u>|_{L^2(V)^3} \leqslant c\ |\varepsilon(u)|_{L^2(V)^9_s} .$$

3.2 Prove that the solution $\varepsilon(u^\star)$ of (13) has the following variational property : $\varepsilon(u^\star)$ minimizes among all $\varepsilon(\vec{u}^\star)$ in $\varepsilon(V_o)$ the microscopic elastic energy

$$<(\varepsilon(\vec{u}^\star) + E) : a : (\varepsilon(\vec{u}^\star) + E)>$$

and prove that the minimum is $E : a^{hom} : E$. Using the inclusions $\tilde{V} \subset V_{per} \subset \hat{V}$ prove that for every E in $\mathbb{R}^9_s$

$$E : \hat{a}^{hom} : E \leqslant E : a^{hom}_{per} : E \leqslant E : \tilde{a}^{hom} : E .$$

3.3 Prove that the long memory entering (26) is fading

$$(\exists\, c > 0)(\exists\, k > 0) \qquad |J(t)| \leqslant c\ e^{-kt}$$

3.4 Consider a KELVIN-VOIGT's material

$$\sigma = a : \varepsilon(u) + b : \varepsilon(\dot{u})$$

Prove that the homogenized constitutive law is :

$$\Sigma(t) = a^{hom} : E(t) + \int_0^t K(t-s) : \dot{E}(s)ds + b^{hom} : \dot{E}(t) \quad (\text{cf. } ^{14})$$

where K is a kernel which you will specify.

Part 4 4.1 Consider a stratified two phase composite material, infinite and homogeneous in the directions (y_1, y_2) and periodically heterogeneous in the direction (y_3). Show that (cf. [24])

$$P^{hom} = \{\sigma \mid \Sigma = c_1 \Sigma^1 + c_2 \Sigma^2 \; ; \quad \Sigma^1 \in P^1 \; , \Sigma^2 \in P^2 \, , \Sigma^1_{i3} = \Sigma^2_{i3}\}$$

where P^1 and P^2 are the yield locus of the phases 1 and 2. Assume moreover that the two constituents are Tresca's materials

$$P^i = \{\sigma \mid \sup_{k,\ell} |\sigma_k - \sigma_\ell| \leqslant 2k^i\}$$

where σ_k denote the principal stresses of σ. Restricting your attention to 2 dim problems, show that

$$P^{hom} = \{\Sigma \mid \sup_{k,\ell} |\Sigma_k - \Sigma_\ell| \leqslant 2C(\alpha)\}$$

where α is the angle between Oy_1 and the direction of the major principal stress.

Part 5 5.1 In case of a MAXWELL's viscoelastic material relate the result (60)(61) with the homogenized constitutive law established in section 3. Compute the macroscopic dissipation with the help of (2), and show that it was not possible to derive it directly from (26).

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