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Approach by Homogenization of Some Linear and Nonlinear Problems in Solid Mechanics

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Abstract. – This work presents a few applications of the homogenization theory. Homogenization is the process which allows to derive macroscopic constitutive equations from microscopic ones. The method is applied to elastic, elastic damaged, elastic viscoplastic and plastic damaged media. A good agreement with existing bounds for elastic composites is shown. Numerical results for elastic damage are presented. A general method of deriving macroscopic constitutive laws for dissipative materials is proposed.

Résumé. – ***Approche par Homogénéisation de Quelques Problèmes Linéaires et Non-Linéaires en Mécanique des Solides.*** – On présente dans ce travail des applications de la méthode d'homogénéisation. Cette méthode permet, pour des milieux périodiques ou quasi-périodiques, de déduire les lois de comportement de certains milieux fortement hétérogènes à partir de leur loi microscopique. La méthode est appliquée successivement aux milieux élastiques (composites), élastiques endommagés (microvides), élastiques viscoplastiques et plastiques endommagés. Dans le cas de milieux élastiques, elle est comparée à des bornes existantes. Des résultats numériques concernant les milieux élastiques endommagés sont présentés. Un procédé général d'homogénéisation de processus dissipatifs standard est proposé.

1. Homogenization in the large

Homogenization is the process in which a homogeneous medium is substituted for a highly heterogeneous one. This homogeneous medium must behave

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in a manner similar to that of the given heterogeneous body, when the phenomena of interest are measured on a scale which is very large with respect to the size of the heterogeneities. This homogenization process has taken various forms since the early works of Maxwell. In the Mechanics of Solids, the contributions of Hashin-Shtrikman [11], Hill [12], Kröner [13] are the best known. More recently, this theory has been fairly developed on its mathematical side in several studies concerning mainly periodic media (Sanchez [24], Bensoussan-Lions-Papanicolaou [3], Duvaut [9]).

1.1. Why quasi-periodic media ?

Periodic media are quite commonly encountered in the human buildings. But their study can be further motivated by the following remark : when dealing with statistically homogeneous media, one can only obtain bounds on the material properties. The simplest bounds depend on the ratio of the constituents, while more elaborate theories take into account anisotropy (through correlations of higher order [13]). These bounds are exact in the sense that, for each of them, a material configuration, respecting the data and whose homogeneous properties are given by the bound, can be found. However in some cases involving a mixture of quite different constituents these bounds can be very deceiving. The same remark holds for highly oriented media, the anisotropy of which cannot be well taken into account by the most convenient theories.

On the other hand if we assume that the medium is quasi-periodic we seriously restrict the generality of the geometry. Despite this drawback, for a given geometry of the heterogeneities (that are distributed in a quasi-periodic manner) we obtain a unique set of material properties for the homogeneous body by the so called "homogenization theory". This theory applying takes into account the exact geometry of the heterogeneities and hence, accounts for anisotropy.

Moreover, as we will see later, if we deal with a statistically homogeneous body, for which classical bounds give an admissible result (i.e. the upper and lower bounds are rather close to one another), the homogenization method, applied with the assumption that the body is ideally periodic, gives a satisfying result, which lies between the two bounds, and which is often close to the experimental results (cf. § 2).

Finally the homogenization method can be justified in a sense which will appear more clearly in the sequel. We shall apply this method in various situations : the study of the elastic properties of composites (§ 2), the construction of a model of damage for elastic media (§ 3), elasto-viscoplastic or elasto-plastic composites (§ 4), damaged elasto-plastic media (§ 5).

1.2. Notations of the problem

The body under consideration occupies a domain Ω in R^3. It exhibits an ϵ Y periodic structure in the following sense : $Y = \prod_{i=1}^{3}]0, Y_i[$ is the basic cell. It is made from different constituents ; one of these constituents can be eventually absent (i.e. void). ϵ is the similarity factor between the basic cell Y and the elementary cell ϵY which generates Ω by periodicity (see Fig. 1). A point in the homogeneous body is referenced by its coordinates $x = (x_i)$ (macroscopic coordinates), a point in Y is referenced by its coordinates $y = (y_i)$ (microscopic coordinates), while a point of the heterogeneous body is referenced by (x , y) x denoting the rough position of the point, and y denoting the position of the point in the cell located "under x". If Y is a fixed set, the medium is exactly periodic ; if Y depends on the slow variable x in a smooth manner the medium is quasi-periodic.

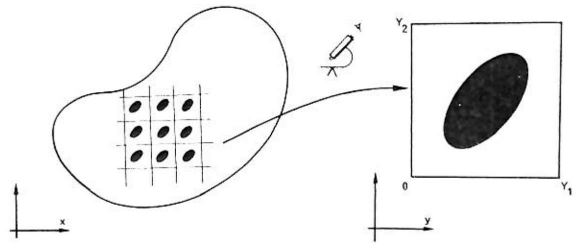


Figure 1. – *The heterogeneous body. The basic cell.*

We shall develop in the sequel one form of the homogenization method : the most intuitive one is the mean values (or averaging) method. It is similar in principle to what happens to a "short sighted" person who has just a rough (or averaged) perception of a more detailed landscape. In a heterogeneous body, the stress and strain fields (or any other field of state variable) are very oscillating quantities. We are only interested in global information : in a first stage, macroscopic quantities are derived from microscopic ones. For most physical variables (but not for all of them) this procedure consists simply in averaging on an elementary cell. The second stage consists in relating these macroscopic quantities, using the microscopic constitutive law and the quasi-periodic geometry : this micromechanics study leads to the homogenized law.

1.3. Another method

Let us just make a few remarks about the convergence method, which is the way of justifying all the developments : there exists in the problem a small parameter ϵ related to the size of the heterogeneity. We can (formally) expand the heterogeneous medium (geometry + material properties) in terms of powers of ϵ :

$$\Omega^{\epsilon} = \Omega^{0} + \epsilon\, Y \tag{1}$$

where Ω^0 is the homogeneous medium (geometry + material properties) we are looking for. In the real body Ω^ϵ, which corresponds to a finite non zero value of ϵ, the stress and strain fields are denoted by $\boldsymbol{\sigma}^\epsilon$ and e^ϵ. The idea of the method is to substitute for fields their limits when ϵ tends to 0, which is reasonable, since ϵ is a small parameter.

$$\boldsymbol{\sigma}^{0} = \lim_{\epsilon\to 0} \boldsymbol{\sigma}^{\epsilon} \qquad e^{0} = \lim_{\epsilon\to 0} e^{\epsilon}$$

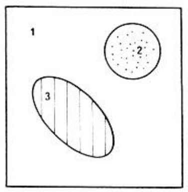
The homogenized law relates the two limits (and eventually other state variables).

In most cases, the two approaches (averaging and convergence) are equivalent. Although the second one is rigorous and justifies all the results obtained, we shall prefer the first one, which bears some resemblance to more classical methods [12] and which has a more physical background. The reader must be aware that in certain circumstances this method is only formal.

2. Homogenization for elastic composites

2.1. The method

We consider here a composite material with two or more constituents (no holes). We shall restrict our attention to periodic media, the extension to quasi-periodic media being straightforward. Because of the periodicity of the geometry, the constitutive law of the whole body is completely defined by the data of the microscopic constitutive law on the basic cell and by the geometry of the heterogeneities.



$$A_{ijkh}(y) = \begin{cases} A^{1}_{ijkh} & \text{if } y \in 1 \\ A^{2}_{ijkh} & \text{if } y \in 2 \\ A^{3}_{ijkh} & \text{if } y \in 3 \end{cases} \tag{3}$$

Figure 2. – *The basic cell and its constituents.*

The constituents are assumed to be elastic (not necessarily isotropic) and perfectly bonded. All perturbations are assumed to be infinitesimal. The microscopic constitutive law is :

$$e_{ij}(\mathbf{u}(x,y)) = A_{ijkh}(y)\, \sigma_{kh}(x,y), \tag{4}$$

$\mathbf{e}(\mathbf{u}(x,y))$ and $\boldsymbol{\sigma}(x,y)$ are the microscopic strain and stress fields. Our reasoning is based on the following principle : at a macroscopic point x, the macroscopic fields of strain, stress and free energy are the average on the cell of the microscopic quantities. For a given function $f(x,y)$ we define :

$$f^{0}(x) = \frac{1}{|Y|} \int_{Y} f(x,y)\, dy = \langle f(x,y) \rangle. \tag{5}$$ [1]

Thus :

$$\boldsymbol{\sigma}^{0}(x) = \langle \boldsymbol{\sigma}(x,y) \rangle, \quad \mathbf{e}^{0}(x) = \langle \mathbf{e}(\mathbf{u}(x,y)) \rangle. \tag{6}$$

Our aim is to obtain the constitutive law relating $\boldsymbol{\sigma}^0$ and $\mathbf{e}^0$, from (4) and the periodicity of the geometry. Equivalently we can try to identify the coefficients A^{eff}_{ijkh} of a quadratic form in $\boldsymbol{\sigma}^0$ [2] :

$$Q^{\text{eff}}(\boldsymbol{\sigma}^{0}, \boldsymbol{\tau}^{0}) = A^{\text{eff}}_{ijkh}\, \sigma^{0}_{ij}\, \tau^{0}_{kh}. \tag{7}$$

The elastic energy in the stress state $\boldsymbol{\sigma}^0$ is :

$$w_*^{\text{eff}}(\boldsymbol{\sigma}^{0}) = \frac{1}{2} Q^{\text{eff}}(\boldsymbol{\sigma}^{0}, \boldsymbol{\sigma}^{0}) = \frac{1}{2} A^{\text{eff}}_{ijkh}\, \sigma^{0}_{ij}\, \sigma^{0}_{kh}. \tag{8}$$

According to the averaging principle we have :

$$Q^{\text{eff}}(\boldsymbol{\sigma}^{0}, \tau^{0}) = \langle Q(y.\ \boldsymbol{\sigma}(y),\ \tau(y)) \rangle = \langle A_{pqrs}(y)\, \sigma_{pq}(y)\, \tau_{rs}(y) \rangle \tag{9}$$

where $\boldsymbol{\sigma}(y)$ and $\boldsymbol{\tau}(y)$ are the microscopic stress states related to $\boldsymbol{\sigma}^0$ and $\boldsymbol{\tau}^0$. We consider the following elementary stress states $\mathbf{T}_{ij}$:

$$T_{ij|pq} = \frac{1}{2} (\delta_{ip}\, \delta_{jq} + \delta_{iq}\, \delta_{jp}). \tag{10}$$ [3]

Then :

$$A^{\text{eff}}_{ijkh} = Q^{\text{eff}}(\mathbf{T}_{ij}, \mathbf{T}_{kh}). \tag{11}$$

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[1] $|Y|$: volume of Y.

[2] We omit x for the sake of simplicity.

[3] To avoid confusion with partial derivative we will denote the component (pq) of the tensor $\mathbf{T}_{ij}$ by $T_{ij|pq}$.

From (8) we deduce that it is sufficient to know the microscopic stress states $\mathbf{C}_{ij}(y)$ due to the six macroscopic stress states $\mathbf{T}_{ij}$. This can be done with the help of the constitutive law (cf. 12)), the equilibrium equations (13) (no concentrated forces) and the condition of average (14) :

$$\mathbf{A}(y)\,\mathbf{C}_{ij}(y) = \mathbf{e}(\mathbf{u}_{ij}) \quad \text{i.e. } A_{pqrs}\, C_{ij|rs} = e_{pq}(\mathbf{u}_{ij}) \text{ in } Y^1, \tag{12}$$

$$\operatorname{div} \mathbf{C}_{ij} = 0 \quad \text{i.e. } \frac{\partial}{\partial y_q} C_{ij|pq} = 0 \text{ in } Y, \tag{13}$$

$$< \mathbf{C}_{ij} > = \mathbf{T}_{ij} \quad \text{i.e. } < C_{ij|pq} > = T_{ij|pq}. \tag{14}$$

But the so obtained problem is not well posed (boundary conditions are missing) : the periodicity condition is now used in an essential manner. At a certain distance of the boundary, the strain and stress fields are certainly periodic. So, we look for a stress field $\mathbf{C}_{ij}$ satisfying :

$$\mathbf{C}_{ij}\,\mathbf{n} \text{ opposite on opposite sides of } Y \text{ (with outer normal n)} \tag{15}$$

and for a *displacement field* $\mathbf{u}_{ij}$ *inducing a periodic strain field.* It can easily be checked that such a displacement field is either periodic, or linear, or the sum of two such fields (we disregard rigid displacements) :

$$\mathbf{u}_{ij} \in DP(Y) \tag{16}$$

$$DP(Y) = \{\mathbf{u} = (u_i)_{1\leqslant i \leqslant 3},\, u_i = E_{ij}\, y_j + v_i,\, E_{ij} = < e_{ij}(\mathbf{u}) >, v_i \text{ periodic}\}$$

The problem (12) (13) (14) (15) (16) is not a classical boundary value problem in Y, since the boundary conditions are of a periodic type. However, it is a well posed problem and it can be shown [26] that it admits a solution $(\mathbf{C}_{ij}, \mathbf{u}_{ij})$, $\mathbf{C}_{ij}$ being unique while $\mathbf{u}_{ij}$ is defined up to a rigid displacement of Y. By virtue of (9) and (11), we get :

$$A^{\text{eff}}_{ijkh} = < A_{pqrs}(y)\, C_{ij|pq}(y)\, C_{kh|rs}(y) >, \tag{17}$$

which are the coefficients of the homogenized law :

$$e^0_{ij} = A^{\text{eff}}_{ijkh}\, \sigma^0_{kh}\,. \tag{18}$$

2.2. Alternative approaches

In the previous sections we derived the constitutive law with the help of equation (9). The equivalent homogeneous body was thus found by an *equivalence in energy.* The same constitutive law can be derived by an *equivalence in stress* (it is the so called dual method) or by an *equivalence in strain* (it is the so called primal method).

[1] For a vector $\mathbf{u}$ we denote $e_{pq}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_p}{\partial y_q} + \frac{\partial u_q}{\partial y_p}\right)$

2.2.1. Equivalence in stress

Taking the average in (4) yields :

$$e^0_{ij} = < A_{ijpq}(y)\, \sigma_{pq}(y) > \tag{19}$$

The problem is then reduced to expressing $\boldsymbol{\sigma}(y)$ as a function of its average $\boldsymbol{\sigma}^0$, which can be done very simply by solving the following elasticity problem on the basic cell Y :

$$\left.\begin{array}{l} \boldsymbol{\sigma}^0 \textit{ being given, find } \boldsymbol{\sigma}(y) \textit{ and } \mathbf{u}(y) \textit{ such that :}\\ \mathbf{e}(\mathbf{u}) = \mathbf{A}(y)\,\boldsymbol{\sigma}(y) \text{ in } Y \\ \operatorname{div}\boldsymbol{\sigma}(y) = 0 \text{ in } Y \\ < \boldsymbol{\sigma}(y) > = \boldsymbol{\sigma}^0 \\ \boldsymbol{\sigma}(y)\,\mathbf{n}(y) \textit{ opposite on opposite sides of } Y \\ \mathbf{u}(y) \in DP(Y) \end{array}\right\} \tag{20}$$

Since the problem (20) is linear we can split the data $\boldsymbol{\sigma}^0$ and the solution $\boldsymbol{\sigma}(y)$ in the following manner :

$$\sigma^0_{pq} = T_{kh|pq}\, \sigma^0_{kh} \tag{21}$$

$$\sigma_{pq}(y) = C_{kh|pq}(y)\, \sigma^0_{kh} \tag{22}$$

$\mathbf{C}_{kh}$ is defined by (12) (13) (14) (15) (16) which is just the problem (20) with $\boldsymbol{\sigma}^0 = \mathbf{T}_{kh}$. Then (19) yields, since $\boldsymbol{\sigma}^0$ is constant :

$$e^0_{ij} = < A_{ijpq}(y)\, C_{kh|pq}(y) > \sigma^0_{kh} \tag{23}$$

and we find the following homogenized law :

$$A^{\text{eff}}_{ijkh} = < A_{ijpq}(y)\, C_{kh|pq}(y) > \tag{24}$$

The equivalence between the two formulas (24) and (17) follows easily from a variation of Hill's lemma :

Lemma : Let $\mathbf{u}$ *be an element of DP(Y) and* $\boldsymbol{\tau}$ *be a microscopic stress state such that :*

$\operatorname{div}\boldsymbol{\tau}(y) = 0$ *in Y,* $\boldsymbol{\tau}\,\mathbf{n}$ *opposite on opposite sides of Y, then :*

$$< \boldsymbol{\tau}\,\mathbf{e}(\mathbf{u}) > = <\boldsymbol{\tau}><\mathbf{e}(\mathbf{u})> \tag{25}$$

Applying this lemma with $\mathbf{u} = \mathbf{u}_{kh}$ and $\boldsymbol{\tau} = \mathbf{C}_{ij}$ yields the desired result.

Remark 1 – The fourth order tensor **C** with components $C_{ij|kh}$ is called the “elastic stress localization tensor” since by virtue of (22) it gives the microscopic stress as a function of the macroscopic stress.

2.2.2. Equivalence in strain :

We can invert the constitutive law (4)

$$\sigma_{ij}(y) = a_{ijpq}(y)\, e_{pq}(u(y)) \tag{26}$$

Then we obtain by averaging :

$$\sigma^0_{ij} = \langle a_{ijpq}(y)\, e_{pq}(u(y)) \rangle \tag{27}$$

The problem is now reduced to expressing the microscopic strain $\mathbf{e}(\mathbf{u}(y))$ in terms of its average $\mathbf{e}^0$. This is done by solving the following problem :

$\mathbf{e}^0$ being given, find $\boldsymbol{\sigma}(y)$ and $\mathbf{u}(y)$ such that :

$$\left.\begin{array}{ll} \boldsymbol{\sigma}(y) = a(y)\, \mathbf{e}(\mathbf{u}(y)) & \text{in } Y \\ \operatorname{div} \boldsymbol{\sigma}(y) = 0 & \text{in } Y \\ \langle \mathbf{e}(\mathbf{u}(y)) \rangle = \mathbf{e}^0 & \\ \boldsymbol{\sigma}\, \mathbf{n} \text{ opposite on opposite sides of } Y & \\ \mathbf{u} \in DP(Y) & \end{array}\right\} \tag{28}$$

The problem (28) is linear ; the data $\mathbf{e}^0$ can be split into elementary data :

$$e^0_{pq} = e^0_{kh}\, T_{kh|pq}$$

and the solution $\mathbf{u}(y)$ of (28) can be split into elementary solutions :

$$\mathbf{u}(y) = e^0_{kh}\, \mathbf{w}_{kh}(y) \tag{29}$$

where $\mathbf{w}_{kh}$ denotes the displacement solution of (28) with $\mathbf{e}^0 = \mathbf{T}_{kh}$. Then :

$$e_{pq}(\mathbf{u}(y)) = e^0_{kh}\, e_{pq}(\mathbf{w}_{kh}(y)) \tag{30}$$

By virtue of (18) we obtain :

$$\sigma^0_{ij} = \langle a_{ijpq}(y)\, e_{pq}(\mathbf{w}_{kh}(y)) \rangle\, e^0_{kh}, \tag{31}$$

and the homogenized constitutive law is the defined by :

$$a^{\text{eff}}_{ijkh} = \langle a_{ijpq}(y)\, e_{pq}(\mathbf{w}_{kh}(y)) \rangle \tag{32}$$

Remark 2 – It can be proved that $\mathbf{a}^{\text{eff}}$ and $\mathbf{A}^{\text{eff}}$ are inverse tensors. By the variation of Hill's lemma it can be shown that :

$$a^{\text{eff}}_{ijkh} = \langle a_{pqrs}(y)\, e_{pq}(\mathbf{w}_{kh}(y))\, e_{rs}(\mathbf{w}_{ij}(y)) \rangle \tag{33}$$

Remark 3 – The fourth order tensor **c** with components $e_{ij}(\mathbf{w}_{kh})$ is called the “elastic strain localization tensor” by virtue of (30).

2.3. Anisotropy

2.3.1. Theoretical

For an arbitrary heterogeneity the homogenized law is completely anisotropic (a fortiori if the constituents are already anisotropic). Its definition requires the solving of 6 elastic problems corresponding to the 6 elementary macroscopic states of stress (in the stress approach) or of strain (in the strain approach). The solutions of these problems obviously depend on the shape of the heterogeneities : this leads to the macroscopic anisotropy.

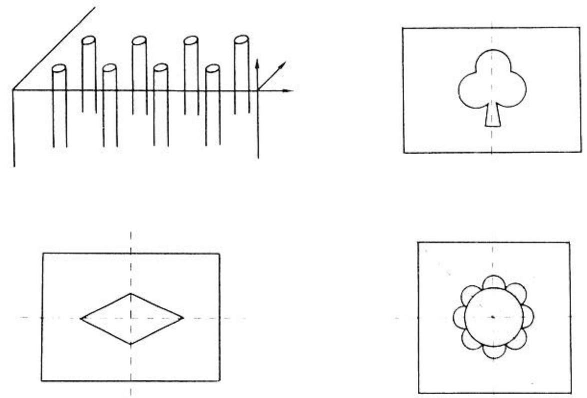

Figure 3. – *Symmetries of the basic cell.*

However the degree of anisotropy can be reduced because of the symmetries of the basic cell (geometrical + material symmetries). If we consider for instance a fibrous monodirectional (Fig. 3) material F. Léné [16] has established the following points :

a) Cross section of arbitrary shape. – 13 coefficients are required to define the constitutive law which has the following form :

$$\begin{pmatrix} \sigma^0_{11} \\ \sigma^0_{22} \\ \sigma^0_{33} \\ \sigma^0_{12} \\ \sigma^0_{23} \\ \sigma^0_{13} \end{pmatrix} = \begin{pmatrix} Q_{1111} & Q_{1122} & Q_{1133} & 0 & 0 & 2Q_{1131} \\ & Q_{2222} & Q_{2233} & 0 & 0 & 2Q_{2231} \\ & & Q_{3333} & 0 & 0 & 2Q_{3331} \\ & & & 2Q_{1212} & 0 & 0 \\ & & & & 2Q_{2323} & 2Q_{2331} \\ & \text{sym.} & & & & 2Q_{1313} \end{pmatrix} \begin{pmatrix} e^0_{11} \\ e^0_{22} \\ e^0_{33} \\ e^0_{12} \\ e^0_{23} \\ e^0_{13} \end{pmatrix}$$

b) One symmetry. – If the cross section possesses a symmetry with respect to a coordinate axis the number of coefficients reduces to 9 (orthotropy) and the elastic problems can be solved on a half cell.

$$Q_{1131} = Q_{2231} = Q_{3331} = Q_{2331} = 0.$$

c) Two symmetries. – If the cross section of the basic cell possesses two symmetries with respect to the two axis of coordinates the number of coefficients is still 9 but the elastic problems can be solved on a quarter cell, with ordinary boundary conditions (this avoids the difficulties due to the periodicity conditions).

d) Three symmetries. – If the cross section of the basic cell possesses 3 symmetries, namely 2 symmetries with respect to the coordinate axis and a symmetry with respect to a diagonal axis, then the number of coefficients reduces to 6.

$$Q_{1122} = Q_{1133} \,,\; Q_{1313} = Q_{2323} \,,\; Q_{1111} = Q_{2222}$$

It should be noted that, if the basic cell possesses an hexagonal symmetry, then the macroscopic constitutive law is transversely isotropic, and requires the determination of 5 coefficients. This is the case if the fibers are circular and arranged at the vertices of a regular triangular net.

2.3.2. Numerical experiments

a) Some hints for computations :

The most manageable approach from a computational standpoint seems to be the strain approach. 6 displacements vectors $\mathbf{w}_{ij}$ solutions of (28) with $\mathbf{e}^0 = \mathbf{T}_{ij}$, have to be determined. But, according to the decomposition of an element of DP(Y), we have :

$$\mathbf{w}_{ij} = \mathbf{T}_{ij}\, y + \chi_{ij} \qquad (34)$$

where χ_{ij} is a vector with periodic components. Using the variation of Hill's lemma it can be shown that χ_{ij} is the solution of the following variational problem :

Find a vector χ_{ij} with periodic components such that:

$$\int_Y a(y)\, e(\chi_{ij})\, e(\phi)\, dy = -\int_Y a(y)\, \mathbf{T}_{ij}\, e(\phi)\, dy \qquad (35)$$

for every vector ϕ with periodic components.

The homogenized coefficients are given by (32) with due account of the splitting of $\mathbf{w}_{ij}$.

b) Some examples :

i) We have tested the method on a case for which experimental results are known for a statistically homogenous medium. The body under investigation is a (monodirectionnaly reinforced) boron epoxy composite with the following characteristics :

$$E_f = 60.10^6 \text{ PSI} \qquad \nu_f = 0.20$$
$$E_m = .06.10^6 \text{ PSI} \qquad \nu_m = 0.35$$

The direction of the fibers is y_3. The transverse modulus E_1, the Poisson coefficient ν_{31} are plotted on Fig. 4. For the homogenization theory a very idealized situation has been chosen : the fibers are assumed to have a circular cross section, and are arranged at the vertices of a square net. As it can be seen the agreement with experimental data and with previous bounds theories (for statistically homogeneous media) is quite good.

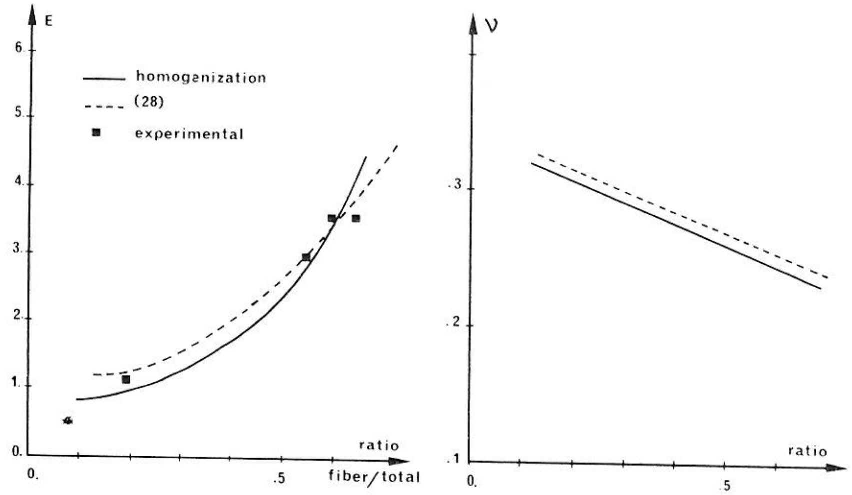

Figure 4. – *Comparison Homogenization/experimental data.*

ii) We have studied the effect of the shape of the heterogeneity on the homogenized properties, when the ratio fiber-matrix is fixed. This has been done for an elliptic shape, with constant area. It shows how easily the homo-

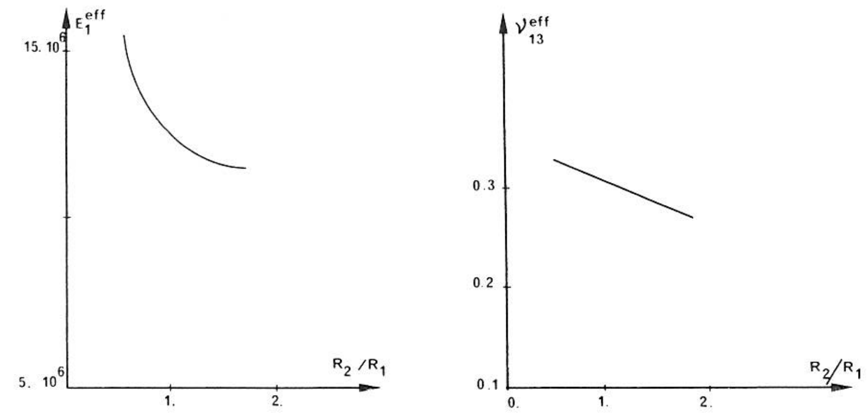

Figure 5. – *Effect of the shape of the fiber : elliptic cross section.*

genization theory accounts for anisotropy : the computer programs are the ones of the first example.

3. Model of damaged material

3.1. Description of the model

A material submitted to cyclic or heavy loadings is damaged : its elastic properties are degraded during the loading process. This macroscopic evidence reflects a microscopic process : on the microscopic scale, voids and cracks are initiated and grow during the material history. This microscopic process has undoubtedly been analyzed a long time ago by the pioneers of damage theory (Kachanov & Rabotnov) but its complexity has limited its use to qualitative explanations. Since the measurable quantities are essentially macroscopic, the studies on damage are generally oriented towards the introduction of macroscopic variables describing the phenomenon. As long as one is concerned with unidirectional problems, the Kachanov & Rabotnov theory is quite satisfactory. It is based on the notion of effective stress :

$$\tilde{\sigma} = \sigma/(1-D) \qquad \text{(D measures the damage)} \tag{36}$$

which yields for the elastic moduli :

$$E^{eff} = E(1-D) \tag{37}$$

Patented authors [15] agree that the evolution of the damage parameter D is correctly described by a Norton law :

$$\dot{D} = \alpha(\tilde{\sigma})^n \tag{38}$$

This theory has been successfully generalized [6] [15] to the case of a three dimensional isotropic damage for which only one parameter is needed. However the real phenomenon seems to be anisotropic and the propagation of damage takes place along preferential directions related to the stress state. Thus the question of the definition of damage, and of the finding of the evolution laws for the damage parameters naturally arises.

A few authors [6] [7] [23] [21] have tried a macroscopic approach to this problem. It appears that the definition of D is not straight-forward. The theories based on equivalences in strain, in stress or in energy do not compare. Others authors have resorted to micro-mechanical considerations to derive a macroscopic model [1] [4] [22] : we believe however that they did not use the adequate tool for this scale change. The "good" tool is the homogenization theory.

We suppose here that the damaged medium can be modelled as a continuum with microdefects distributed in a quasi-periodic manner. The situation is comparable to the one of § 2, but the micro-defects are substituted to micro-heterogeneities in the basic cell. We assume that the size of micro-defects can vary continuously on the macroscopic scale. This implies that the medium is quasi-periodic for the microscopic scale, since the dependence of a quantity on the macroscopic variable is not perceived at the microscopic level. We assume that the basic micro-defects have a known simple geometry which can be characterized by a finite number of geometric parameters D_1, $D_2, \ldots\ldots, D_n$. These parameters (or any one to one function of them) will be called *the damage parameters.*

Examples (cf. Fig. 6).

The homogenization theory exposed in § 2 applies (the results are unchanged, the arguments differ slightly). It yields the macroscopic properties of the damaged medium as functions of the damage parameters $D_1, \ldots\ldots, D_n$. Let us introduce the following notations : for a macroscopic point x we denote by :

- $D(x)$ the array of the damage parameters $D_1(x) \ldots\ldots D_n(x)$
- $B(D(x))$ the micro-defect (s) in the microscopic cell lying "under" x, the geometry of which is fixed by the data of $D(x)$

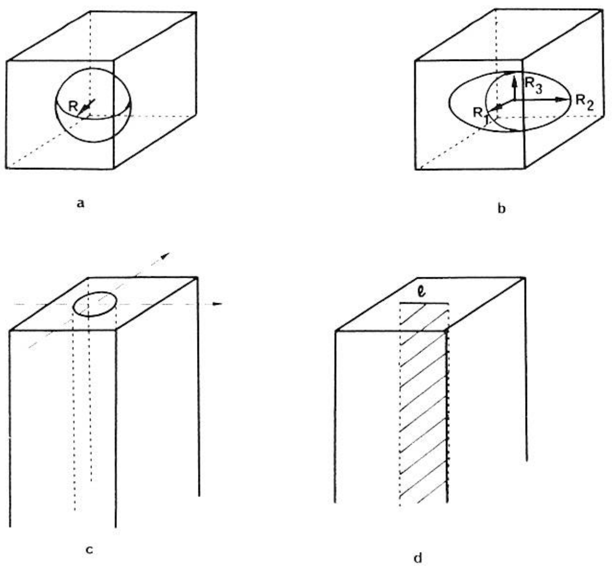

Figure 6. – *Microdefects.*
a) Spherical microvoid : $D = R$ or $D = \frac{4}{3}\pi R^3 / |Y|$
b) Ellipsoidal microvoid : $D_1 = R_1$, $D_2 = R_2$, $D_3 = R_3$ or $D_i = \frac{4}{3}\pi R_i^3 / |Y|$
c) Cylindrical microvoid with an elliptic cross section : $D_1 = R_1$, $D_2 = R_2$
d) Plane microcrack : $D_1 = \ell$

• $\mathbf{C}^D_{ij}$ (x) the stress localization tensor defined as follows:

$$\left\{\begin{array}{ll} A(y)\,\mathbf{C}^D_{ij} = \mathbf{e}(\mathbf{u}^D_{ij}) & \text{in } \; Y^*(x) = Y - B(D(x)) \\ \operatorname{div} \mathbf{C}^D_{ij} = 0 & \text{in } \; Y^*(x) \\ \mathbf{C}^D_{ij}\,\mathbf{n} = 0 \text{ on } \partial B \text{ (boundary of } B(D(x))) & \\ < \mathbf{C}^D_{ij} > = \mathbf{T}_{ij} & \\ \mathbf{C}^D_{ij}\,\mathbf{n} \text{ opposite on opposite sides of } Y & \\ \mathbf{u}^D_{ij} \in DP(Y) & \end{array}\right\} \qquad (39)$$

The homogenized law is then defined by the following compliance tensor:

$$A^{eff}_{ijkh} = \frac{1}{|Y|}\int_{Y^*(x)} A_{pqrs}\, C^D_{ij|pq}(y)\, C^D_{kh|rs}(y)\, dy \qquad (40)$$

Its inverse tensor can also be useful. We introduce to this effect the strain localization tensor and displacements fields w^D_{ij} defined in a manner similar to that of (28) (30).

$$\left.\begin{array}{ll} \mathbf{s}^D_{ij} = \mathbf{a}(y)\,\mathbf{e}(\mathbf{w}^D_{ij}) & \text{in } Y^* \\ \operatorname{div} \mathbf{s}^D_{ij} = 0 & \text{in } Y^* \\ \mathbf{s}^D_{ij}\,\mathbf{n} = 0 \text{ on } \partial B & \\ \mathbf{s}^D_{ij}\,\mathbf{n} \text{ opposite on opposite sides of } Y & \\ \mathbf{w}^D_{ij} \in DP(Y) & \\ \mathbf{w}^D_{ij} - \mathbf{T}_{ij}\, y \;\; \text{is periodic} & \end{array}\right\} \qquad (41)$$

Then the stiffness tensor is given by :

$$a^{eff}_{ijkh}(D(x)) = \frac{1}{|Y|}\int_{Y^*} a_{pqrs}(y)\, e_{pq}(w^D_{ij}(y))\, e_{rs}(w^D_{kh}(y))\, dy \qquad (42)$$

which generalizes (37) to the three dimensional anisotropic case.

Remark 4 : Let us point out that all the homogenized laws derived for the damaged medium, either by an equivalence in energy or by an equivalence in stress or in strain are *equivalent*. This particular point does not appear clearly in [7] [22] [4].

3.2. *Relations between macroscopic and microscopic quantities. Effective stresses and strains*

Since the stress and strain tensors are not defined in the interior of the defect B (D (x)), the averaging process (6) which yielded the macroscopic quantities from the microscopic ones must be revisited.

In the case of the stress the modification is immediate : since the stress vector vanishes on the boundary of B (D (x)), we can continue the stress tensor by 0 in the interior of the defect. We denote by $\tilde{\boldsymbol{\sigma}}(x, y)$ this continuation and we obtain :

$$\boldsymbol{\sigma}^0(x) = \frac{1}{|Y|}\int_Y \tilde{\boldsymbol{\sigma}}(x, y)\, dy = \frac{1}{|Y|}\int_{Y^*} \boldsymbol{\sigma}(x, y)\, dy \qquad (43)$$

Conversely the microscopic stress state is given, in the case of an elastic medium by the tensor of stress localization defined in (39):

$$\sigma_{rs}(x, y) = C^{D(x)}_{ij|rs}\ \sigma^0_{ij}. \tag{44}$$

We postulate that the *effective stress state is the microscopic one.* Then (44) is a generalization to the three dimensional case of (36). This case clearly illustrates the meaning of the notion of stress localization.

More care is to be applied in the treatment of the strain since the microscopic displacement does not vanish on the boundary of B (D (x)). We will admit that it can be continuated in a regular manner in the inside of B (D (x)). This has been proved by F. Léné [17]. We denote by $\tilde{\mathbf{u}}(x, y)$ this continuation and we obtain:

$$e^0_{ij}(x) = \frac{1}{|Y|}\int_Y e_{ij}(\tilde{\mathbf{u}}(x, y))\,dy = \frac{1}{|Y|}\int_{\partial Y}\frac{1}{2}(u_i n_j + u_j n_i)\,ds^1 \tag{45}$$

Conversely the microscopic strain is obtained from the macroscopic one by means of the tensor of strain localization:

$$e_{rs}(\mathbf{u}(x, y)) = e_{rs}(\mathbf{w}^{D(x)}_{ij}(y))\, e^0_{ij} \tag{46}$$

We admit that the effective strain is the microscopic strain. It is given by (46) in the solid part of the cell and by the above mentionned continuation, in B (D (x)).

We remark that the variation of Hill's lemma still holds:

Let **u** *be an element of DP (Y) and* τ *be such that*

div $\tau = 0$ in Y*, $\tau \cdot \mathbf{n} = 0$ *on* ∂B, $\tau \cdot \mathbf{n}$ *opposite on opposite sides of Y, then*

$$\frac{|Y^*|}{|Y|} < \tau_{ij}(y)\, e_{ij}(\mathbf{u}(y)) >_{Y^*} = \tau^0_{ij}\, e^0_{ij} \tag{47}$$

where

$$\tau^0_{ij} = \frac{1}{|Y|}\int_{Y^*}\tau_{ij}(y)\,dy = \frac{|Y^*|}{|Y|} < \tau_{ij}(y) >_{Y^*}$$

$$e^0_{ij} = \frac{1}{|Y|}\int_{\partial Y}\frac{1}{2}(u_i n_j + u_j n_i)\,ds$$

[1] We assume here that ∂B does not meet ∂Y : then **u** and $\tilde{\mathbf{u}}$ are equal on ∂Y.

The definition of the elements of DP (Y) remains unchanged (cf. (14)) : *but the macroscopic strain* E_{ij} *is not the average on* Y^* *of the microscopic one*; indeed we have :

$$E_{ij} = e^0_{ij} = \frac{1}{|Y|}\int_{\partial Y}\frac{1}{2}(u_i n_j + u_j n_i)\,ds \tag{48}$$

3.3. Analysis of the dissipation

3.3.1. Theoretical analysis

Damage is a phenomenon which dissipates energy. We propose to study the evolution of the energy with the help of thermodynamical arguments. We consider the quasi-static evolution of the *macroscopic* damaged body considered above. We assume that this evolution is isothermal and that strains are infinitesimal. The actual state of the macroscopic body depends on the field of the damage parameters D (x). We shall denote by $\mathbf{u}^D$ the actual *macroscopic* displacement field in the body under the applied loads. The macroscopic free energy is

$$W = W(\mathbf{e}^0(\mathbf{u}^D), D) = \int_\Omega w(\mathbf{e}^0(\mathbf{u}^D(x)), D(x))\,dx \tag{49}$$

where the following notations have been used

- $\mathbf{e}^0$ is as usually the macroscopic strain tensor

$$e^0_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{50}$$

- w is the macroscopic density of free energy

$$w(\mathbf{e}^0(\mathbf{u}), D) = \frac{1}{2}\, a^{\mathrm{eff}}_{ijkh}(D)\, e^0_{ij}(\mathbf{u})\, e^0_{kh}(\mathbf{u}) \tag{51}$$

The two laws of thermodynamics lead to the following equality of powers :

$$\dot{W} + \mathcal{D} = \mathcal{P}_e \tag{52}$$

where $\mathcal{D}$ is the power dissipated by the irreversibilities of the evolution (in our case they are only due to damage), and where $\mathcal{P}_e$ is the power of the external forces. From (49) we derive :

$$\dot{W} = \int_\Omega \left(\frac{\partial w}{\partial e}\, e(\dot{\mathbf{u}}^D) + \frac{\partial w}{\partial D}\dot{D}\right) dx \tag{53}$$

But $\frac{\partial w}{\partial e} = \boldsymbol{\sigma}^D$ and

$$\int_\Omega \boldsymbol{\sigma}^D\, e(\dot{\mathbf{u}}^D)\, dx = \int_{\partial\Omega} \boldsymbol{\sigma}^D \cdot \mathbf{n}\, \dot{\mathbf{u}}^D\, ds - \int_\Omega \operatorname{div} \boldsymbol{\sigma}^D\, \dot{\mathbf{u}}^D\, dx = \mathcal{P}_e \quad (54)$$

Hence (52) yields

$$\mathcal{D} = -\int_\Omega \frac{\partial w}{\partial D}\, \dot{D}\, dx\,. \quad (55)$$

Let us define the thermodynamical damage forces by :

$$F_d(x) = -\frac{\partial w}{\partial D}(x)$$

$$= -\frac{1}{2}\frac{\partial a^{eff}_{ijkh}}{\partial D}(D(x))\, e^0_{ij}(\mathbf{u}^D(x))\, e^0_{kh}(\mathbf{u}^D(x)). \quad (56)$$

Then

$$\mathcal{D} = \int_\Omega F_d(x)\, \dot{D}(x)\, dx\,. \quad (57)$$

3.3.2. Evaluation of F in two simples cases

a) Let us take the simple model of a medium containing spherical microvoids. The damage parameters is the porosity

$$D = \frac{4}{3}\pi R^3 / |Y|$$

we obtain

$$\frac{\partial a^{eff}_{ijkh}}{\partial D} = \frac{|Y|}{4\pi R^2}\frac{\partial a^{eff}_{ijkh}}{\partial R}.$$

With the help of the derivation formula on moving domains we obtain from (35)

$$\frac{\partial a^{eff}_{ijkh}}{\partial R} = \frac{2}{|Y|}\int_{Y-B(D(x))} a_{pqrs}\, e_{rs}(\mathbf{w}^D_{ij})\, e_{pq}\left(\frac{\partial \mathbf{w}^D_{kh}}{\partial R}\right) dy$$

$$-\frac{1}{|Y|}\int_{\partial B(D(x))} a_{pqrs}\, e_{rs}(\mathbf{w}^D_{ij})\, e_{pq}(\mathbf{w}^D_{kh})\, ds\,. \quad (58)$$

From the definition (41) of $\mathbf{w}^D_{kh}$ it follows that $\frac{\partial \mathbf{w}^D_{kh}}{\partial R}$ is a periodic function.

Hence the variation of the Hill's lemma proves that the first term in (58) vanishes ; we are let with

$$\frac{\partial a^{eff}_{ijkh}}{\partial D} = -\frac{1}{4\pi R^2}\int_{\partial B(D(x))} a_{pqrs}\, e_{rs}(\mathbf{w}^D_{ij})\, e_{pq}(\mathbf{w}^D_{kh})\, ds. \quad (59)$$

Since $|\partial B(D(x))| = 4\pi R^2$ (area of a sphere) (56) yields

$$F_d(x) = \left[\frac{1}{2|\partial B(D(x))|}\int_{\partial B(D(x))} a_{pqrs}\, e_{rs}(\mathbf{w}^D_{ij})\, e_{pq}(\mathbf{w}^D_{kh})\, ds\right] e^0_{ij}(\mathbf{u}^D)\, e^0_{kh}(\mathbf{u}^D)\,. \quad (60)$$

Recalling the following property of the tensor of strain concentration

$$e_{rs}(\mathbf{u}(x,y)) = e_{rs}(\mathbf{w}^D_{ij}(y))\, e^0_{ij}(\mathbf{u}^D(x)),$$

(where $\mathbf{u}(x,y)$ is the microscopic displacement in the damaged body) we obtain

$$F_d(x) = \frac{1}{|\partial B(D(x))|}\int_{\partial B(D(x))} \frac{1}{2} a_{pqrs}\, e_{rs}(\mathbf{u}(x,y))\, e_{pq}(\mathbf{u}(x,y))\, ds \quad (61)$$

The relation (61) enables us to identify the damage force associated to spherical microvoids : it is *the surface average of the microscopic elastic energy on the boundary of the void.* A dimensional analysis shows that the damage force has the dimension of a pressure. This result demonstrates that the force which contributes to the propagation of the microdefect (hence of the damage) depends only on the elastic energy on the *boundary* of the defect. This result is satisfactory when it is compared with the one given by the usual theory of damage. If we assume that

$$a^{eff}_{ijkh} = (1-D)\, a_{ijkh}, \quad (62)$$

we obtain a modified damage force

$$\tilde{F}_d = -\frac{\partial w}{\partial D} = \frac{1}{2} a_{ijkh}\, e^0_{kh}(\mathbf{u}^D)\, e^0_{ij}(\mathbf{u}^D)\,. \quad (63)$$

We see that in this theory the force that contributes to the propagation of damage is the elastic energy of the undamaged material in the actual state of strain $\mathbf{e}^0(\mathbf{u}^D)$. We believe that the result given by the homogenization theory is physically more sound.

b) We consider now a medium containing ellipsoidal voids. Our damage parameters are taken to be $D_1 = R_1/Y_1$, $D_2 = R_2/Y_2$, $D_3 = R_3/Y_3$ which are non-dimensional quantities. Then

$$\frac{\partial a^{eff}_{ijkh}}{\partial D_i} = Y_i \frac{\partial a^{eff}_{ijkh}}{\partial R_i} . \qquad (64)$$

In these conditions, it can easily be shown that

$$\frac{\partial a^{eff}_{ijkh}}{\partial R_i} = \frac{1}{|Y|} \int_{\partial B(D(x))} a_{pqrs}\, e_{rs}(w^D_{ij})\, e_{pq}(w^D_{kh}) \frac{\partial \vec{y}}{\partial R_i} \vec{n}\, ds \qquad (65)$$

where $\vec{n}$ denotes the outer vector to the body on $\partial B(D(x))$.

In this model, 3 damage forces corresponding to the three damage parameters are obtained. These three forces govern the propagation of the voids in the three principal directions, which implies that the propagation of damage is anisotropic.

3.4. *Laws of propagation*

As it has been previously seen the dissipation due to damage is

$$\mathcal{D} = \int_\Omega F_d(x)\, \dot{D}(x)\, dx = \sum_{i=1}^{n} \int_\Omega F_{d_i}(x)\, \dot{D}_i(x)\, dx \qquad (66)$$

This dissipation must be positive. A sufficient condition (which is not necessary) is that the propagation of damage be governed by the normality rule :

Normality rule : We assume that there exists a convex function ϕ of $F_d = (F_{d_1}, F_{d_2}, \ldots, F_{d_n})$ such that

$$\dot{D}(x) = \frac{\partial \phi}{\partial F_d}(F_d(x)) \quad \text{i.e.} \quad \dot{D}_i(x) = \frac{\partial \phi}{\partial F_{d_i}}(F_d(x)) \qquad (67)$$

We shall consider some examples.

a) Brittle damage

We define in the space of the F_d forces a closed convex set P_d, and we set

$$\phi(F_d) = \begin{cases} 0 \text{ if } F_d \in P_d \\ +\infty \text{ otherwise} \end{cases} \qquad (68)$$

The meaning of (67) (68) is the following : if the forces F_d lie inside the damage set P_d there is no propagation of damage, whereas if the forces F_d lie on the boundary of P_d there is propagation of damage along a "direction" (in the space of damage parameters) given by the normality law. Forces F_d outside P_d have no physical meaning.

We can rewrite (68) in the following manner. Let $\mathcal{F}_i$ be potential functions defining P_d

$$F \in P_d \iff \mathcal{F}_i\ (F) \leqslant 0 \quad i = 1, \ldots\ldots, n$$

then (67) (68) is equivalent to

$$\begin{cases} \dot{D} = \lambda_i \dfrac{\partial \mathcal{F}_i}{\partial F_d}(F_d) \\ \lambda_i \geqslant 0,\ \lambda_i = 0 \text{ if } \mathcal{F}_i\ (F_d) < 0 \text{ or } \dot{F}_d \dfrac{\partial \mathcal{F}_i}{\partial F_d} < 0. \end{cases} \qquad (69)$$

In the monodimensional case the law (69) reduces to the law of total damage introduced in [5] through a completely different approach.

b) Law with a time effect I

We set

$$\phi(F) = \frac{1}{2\mu} |F - \Pi_{P_d} F|^2, \qquad (70)$$

(67) yields

$$\dot{D} = \begin{cases} 0 \text{ if } F_d \in P_d \\ \dfrac{1}{\mu}(F_d - \Pi_{P_d} F_d) \quad \text{otherwise.} \end{cases} \qquad (71)$$

This law is analogous to the Perzyna's law of viscoplasticity. μ appears as a viscosity parameter. The velocity of the loading and the time scale, are both taken into account.

c) Law with a time effect II

The well known Norton's law can also be used

$$\phi(F) = \frac{\lambda}{n+1}(j(F))^{n+1},$$

where j is the gauge function of the convex set P_d defined by

$$j(F) = \inf\{k \mid F \in k P_d\}.$$

(67) yields

$$\dot{D} = \lambda \frac{\partial j}{\partial F}(F_d)(j(F_d))^n.$$

In order to propose other laws a stronger physical and microscopic background than the author can account for, is required.

Remark 4. – The normality rule (67) has always been a matter of discussion. It has proved its efficiency in other situations : Plasticity, Rupture, etc. . . However in our case its adequacy is debatable because of its strong dependence on the choice of the damage parameters : the evolution law can be normal for a choice of parameters, and not for another one. The reader is referred to [2] [8] for a discussion of similar situations.

3.5. Numerical results

In our model of damage, the numerical studies are threefold :

a) A parametrized study of the coefficients a^{eff}_{ijkh} is undertaken in order to obtain the law $D \to a^{eff}_{ijkh}$ (D). This study requires the homogenization technique, and the resolution of 6 elastic problems on the basic cell for each value of the parameters D. This is the cumbersome part of the model.

b) The computation of the forces of damage : this step is easily performed.

c) A macroscopic study : after the loadings are prescribed and a model of microvoid is chosen, an incremental procedure is performed. We start with a given value of the damage (0 for example), apply a small increment of loads, compute by b) the damage forces, reinitialize the damage parameters with the law (67), recompute the stiffness matrix with the help of a), then apply another increment of loads, etc. . . At the present time the theory is in its stage of infancy [27] and only partial results on parts a) and b) are available.

Part a : *parametrized study of* a^{eff}_{ijkh}

i) We first take a model with a *cylindrical void* of a circular cross section (cf. Fig. 6c). The transverse directions are 1 and 2. The damage parameter is the two dimensional porosity

$$D = \pi R^2 / |Y|$$

We have drawn on Fig. 7 the relevant parameters E_1 and ν_{12}, the transverse Young modulus and the transverse Poisson coefficient. It is worth noting that, for this model, the damage propagates isotropically since only one parameter is considered. It is also worth noting that the transverse Poisson coefficient is not constant. Hence the damage cannot be described by the model (37) of isotropic damage for which ν_{12} is constant and equal to its initial value. Finally it should be noticed that significant changes in the value of the Young's modulus only occurs for large values of the porosity. The author, who is not an experimentalist, is in no position to offer a satisfying explanation to this fact although these values of the porosity seem to be larger than what is actually observed. This is why we have also performed computations on the case of a microcracked body.

ii) The defect under consideration is now a plane crack, and the damage parameter is ℓ / Y_1 where ℓ is the length of the crack. According to [24] the homogenized body has a different stiffness in traction and in compression along the direction y_2. The Young modulus in traction E_2 and the Poisson coefficient ν_{12} have been plotted on Fig. 8. Note that the laws $E_2(D)$ and $\nu_{12}(D)$ seem to be very similar.

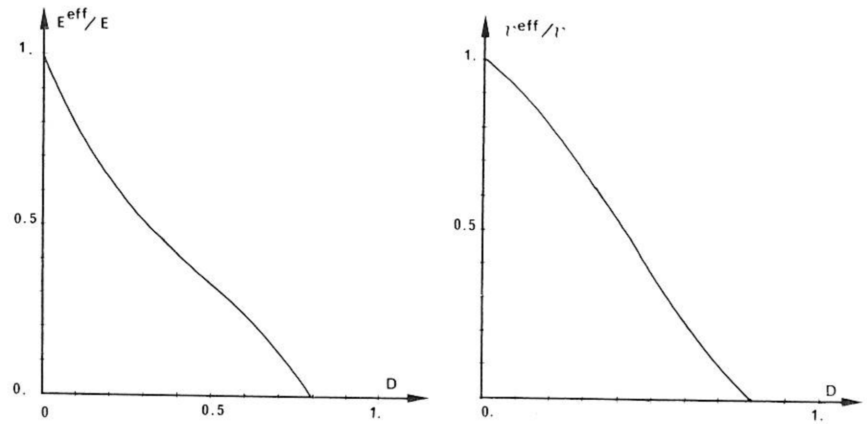

Figure 7. – *Cylindrical microvoid.*

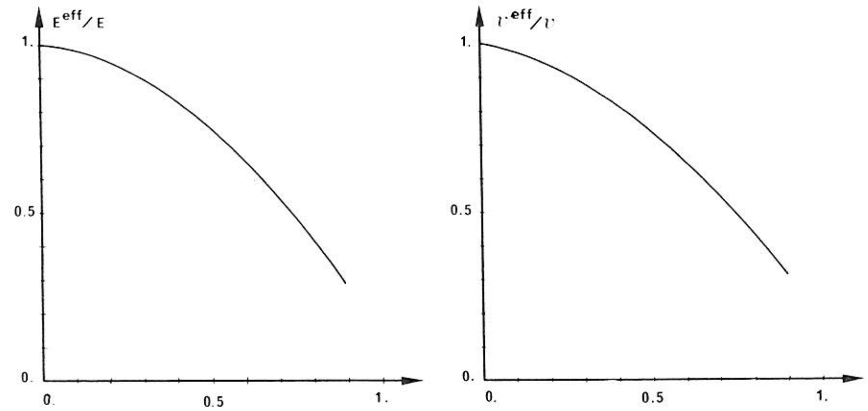

Figure 8. – *Plane crack.*

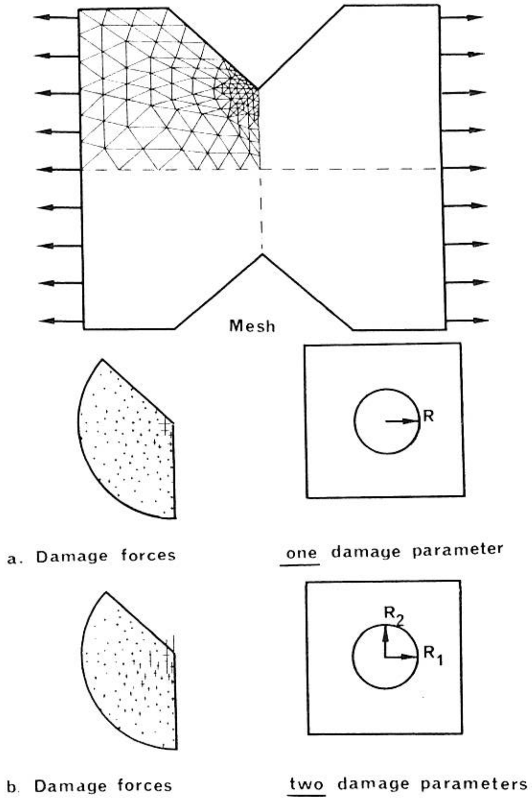

Figure 9. – *Damage forces :*
a) One parameter, b) two parameters of damage.

Part b *: damage forces*

On Fig. 9 we have plotted the damage forces inside a diabolo which has been loaded by imposed displacements at its edges. The computation was performed in plane strain. The Fig. 9a) has been obtained with the model 1) of the cylindrical holes of circular cross section with *one* parameter. The Fig. 9 b) has been obtained with the same model but with *two* parameters, i.e. when considering a circle as an ellipse.

The direction of the forces gives the direction of propagation of damage, i.e. the direction where coalescence of the holes is the most likely to occur. Comparison of both models shows the superiority of the second one, in which anisotropy is present.

4. Homogenization for dissipative materials

4.1. Viscoplastic composites

We now return to the study of composites and we assume that each constituent has a constitutive law constructed after a standard model : the microscopic strain can be split into two parts, the elastic part and the inelastic one. The rate of the anelastic part is given as the derivative of a potential with respect to the stress tensor :

$$\mathrm{e}(\mathbf{u}) = \mathrm{e}^{el} + \mathrm{e}^{an} \tag{72}$$

$$\dot{\mathrm{e}}_{ij}^{an}(x,y) = \frac{\partial \phi}{\partial \sigma_{ij}}(y, \boldsymbol{\sigma}(x,y)) \ . \tag{73}$$

This constitutive law adequately describes a few known cases :

a) Maxwell viscoelasticity

$$\phi(y, \boldsymbol{\sigma}) = \frac{1}{2} B_{ijkh}(y)\, \sigma_{ij}\, \sigma_{kh}, \tag{74}$$

$$\mathrm{e}(\dot{\mathbf{u}}) = \mathbf{A}\dot{\boldsymbol{\sigma}} + \mathbf{B}\boldsymbol{\sigma} \ . \tag{75}$$

b) Perzyna viscoplasticity

$$\phi(y, \boldsymbol{\sigma}) = \frac{1}{2\mu} | \boldsymbol{\sigma} - \Pi_{P(y)} \boldsymbol{\sigma} |^2, \tag{76}$$

where P(y) denotes the elasticity domain at the point y and $\Pi_{P(y)}$ the projection onto P(y) ; then

$$e(\dot{u}) = A\dot{\sigma} + \frac{1}{\mu}(\sigma - \Pi_{P(y)}\sigma) \quad . \tag{77}$$

c) Perfect Plasticity

$$\phi(y,\sigma) = \begin{cases} 0 \text{ if } \sigma \in P(y) \\ +\infty \quad \text{otherwise} \end{cases} \tag{78}$$

(73) yields

$$\begin{cases} (\dot{e}^{an}, \tau - \sigma) \leqslant 0 \quad \forall \tau \in P(y) \\ \sigma(y) \in P(y) \quad . \end{cases} \tag{79}$$

In this first section we shall restrict our attention to regular potentials of dissipation, the derivative of which possesses a Lipschitz property :

$$\left|\frac{\partial \phi}{\partial \sigma}(y,\sigma) - \frac{\partial \phi}{\partial \sigma}(y,\tau)\right| \leqslant C\,|\sigma - \tau| \quad \forall \tau, \sigma \tag{80}$$

This includes the cases a) and b) but excludes c) which will be treated apart.

We construct the homogenized law using the stress approach. An history of macroscopic state of stress $\sigma^0(t)$ being given, the microscopic state of stress $\sigma(y,t)$ and the microscopic displacement field $u(y,t)$ are solutions of

$$\left.\begin{aligned} & A(y)\dot{\sigma}(y,t) + \frac{\partial \phi}{\partial \sigma}(y,\sigma(y,t)) = e(\dot{u}(y,t)) \text{ in } Y \\ & \operatorname{div} \sigma(y,t) = 0 \text{ in } Y \\ & \sigma n \text{ opposite on opposite sides of } Y \\ & \dot{u} \in DP(Y) \\ & \sigma(0) = \sigma_0 \text{ (for the sake of simplicity we shall take } \sigma_0 = 0 \text{ in the sequel)} \end{aligned}\right\} \tag{81}$$

Under the Lipschitz condition (80) on ϕ, the problem (81) admits a solution. In order to obtain the homogenized law we do not immediatly average on Y, but we multiply (81) by the elastic stress concentration tensor C_{rs} and *then* average on Y. In terms of the components we obtain :

$$< A_{ijkh}\, C_{rs|ij}\, \dot{\sigma}_{kh} > + < C_{rs|ij} \frac{\partial \phi}{\partial \sigma_{ij}}(\sigma) > = < e_{ij}(\dot{u})\, C_{rs|ij} > \quad . \tag{82}$$

We note that

$$A_{ijkh}\, C_{rs|ij} = e_{kh}(u_{rs}) \, ; \tag{83}$$

by using the variation of Hill's lemma we get

$$\begin{aligned} < A_{ijkh}\, C_{rs|ij}\, \dot{\sigma}_{kh} > &= < e_{kh}(u_{rs})\, \dot{\sigma}_{kh} > = < e_{kh}(u_{rs}) > < \dot{\sigma}_{kh} > \\ &= A^{eff}_{khrs}\, \dot{\sigma}^0_{kh} \, . \end{aligned} \tag{84}$$

In the same way

$$< e_{ij}(\dot{u})\, C_{rs|ij} > = < e_{ij}(\dot{u}) > < C_{rs|ij} > = \dot{e}^0_{rs} \, . \tag{85}$$

Hence (82) becomes

$$A^{eff}_{khrs}\, \dot{\sigma}^0_{kh} + < C_{rs|ij} \frac{\partial \phi}{\partial \sigma_{ij}}(\sigma) > = \dot{e}^0_{rs} \, . \tag{86}$$

The problem lies in the determination of the term

$$< C_{rs|ij} \frac{\partial \phi}{\partial \sigma_{ij}}(\sigma) > \tag{87}$$

which we wish to express as a function of σ^0. This is impossible to achieve in a direct manner. We split σ into two parts : the one which would occur if the material was completely elastic, and a residual stress tensor :

$$\sigma(y) = C(y)\,\sigma^0 + \sigma^r(y) \tag{88}$$

i.e.

$$\sigma_{ij}(y,t) = C_{ij|rs}(y)\,\sigma^0_{rs}(t) + \sigma^r_{ij}(y,t)$$

σ^r satisfies
$\operatorname{div} \sigma^r = 0, <\sigma^r> = 0, \sigma^r \cdot n$ opposite on opposite sides of Y. Then (87) becomes

$$< C_{rs|ij} \frac{\partial \phi}{\partial \sigma_{ij}}(y, C(y)\,\sigma^0 + \sigma^r(y)) > . \tag{89}$$

We set

$$\Phi(\sigma^0, \sigma^r) = \langle \phi(y, C(y)\sigma^0 + \sigma^r(y)) \rangle , \tag{90}$$

then

$$\frac{\partial \Phi}{\partial \sigma^0_{rs}}(\sigma^0, \sigma^r) = \langle C_{rs|ij}(y) \frac{\partial \phi}{\partial \sigma_{ij}}(y, C(y)\sigma^0 + \sigma^r(y)) \rangle ,$$

and (86) yields

$$A^{eff}_{rskh}\, \dot\sigma^0_{kh} + \frac{\partial \Phi}{\partial \sigma^0_{rs}}(\sigma^0, \sigma^r) = \dot e^0_{rs} . \tag{91}$$

This is the homogenized constitutive law. But we see that the macroscopic data of σ^0 is not sufficient to entirely define the the strain tensor e^0, since we need to know the residual stresses in the cell located "under x". The tensor of residual stresses σ^r satisfies another set of equations, namely

$$\left.\begin{aligned} & A(y)\dot\sigma^r + \frac{\partial \phi}{\partial \sigma}(y, C(y)\sigma^0 + \sigma^r) = e(v^r) \text{ in } Y \\ & \operatorname{div} \sigma^r = 0 \text{ in } Y , \langle \sigma^r \rangle = 0 \\ & v^r \in DP(Y) , \sigma^r \cdot n \text{ opposite on opposite sides of } Y. \end{aligned}\right\} \tag{92}$$

The complete homogenized law is the set of equations (91) (92). These are coupled by the presence of σ^0 and σ^r in both equations. We can rewrite them in a more esthetically pleasing form :

$$\left.\begin{aligned} & A^{eff}\dot\sigma^0 + \frac{\partial \Phi}{\partial \sigma^0}(\sigma^0, \sigma^r) = \dot e^0 \text{ in } \Omega \\ & A\dot\sigma^r + \frac{\partial \Phi}{\partial \sigma^r}(\sigma^0, \sigma^r) = e(v^r) \text{ in } Y \\ & \operatorname{div} \sigma^r = 0 \text{ in } Y , \langle \sigma^r \rangle = 0 \\ & v^r \in DP(Y) , \sigma^r \cdot n \text{ opposite sides of } Y. + \text{initial conditions} \end{aligned}\right\} \tag{93}$$

The homogenized law does not reduce to a single equation *on Ω alone.* It is coupled with another equation on Y. The knowledge of the macroscopic law requires as data the microscopic state variables. It is possible to completely eliminate the microscopic level (i.e. σ^r and v^r). We obtain a functional law where

the whole material history must be taken into account : $\sigma^r(t)$ depends on $\sigma^0(s)$ $0 \leq s \leq t$ (see [24] and [27] for full details in viscoelasticity). But keeping the microscopic level allows for a better insight into the structure of the law (93). If we introduce internal variables, we obtain a "generalized standard law" in the sense of [10]. We shall recall the meaning of this term later on in this text. This kind of constitutive law implies some properties of convexity which are usefull to study the stability or the behavior at large time of such materials.

4.2. Review of the notion of generalized standard materials (GSM)

For such materials the state of the body can be described by the strain tensor e and by other state variables β (we limit ourselves to isothermal transformations). A generalized standard material (G.S.M.) is defined by a density of free energy $W(e, \beta)$ and a density of dissipation $\mathcal{D}(\dot\beta)$. Both are convex functionals of their arguments ; they satisfy

$$\sigma = \frac{\partial W}{\partial e}(e, \beta) \qquad B = \frac{\partial \mathcal{D}}{\partial \beta}(\dot\beta) \tag{94}$$

where we denote by B the thermodynamical force associated to β whose definition is

$$B = -\frac{\partial W}{\partial \beta}(e, \beta) . \tag{95}$$

(94) (95) can be rewritten using the Legendre-Fenchel transforms W_* and Φ of W and $\mathcal{D}$, as

$$e = \frac{\partial W_*}{\partial \sigma}(\sigma, B), \beta = -\frac{\partial W_*}{\partial B}(\sigma, B), \dot\beta = \frac{\partial \Phi}{\partial B}(B) \tag{96}$$

G.S.M.II. – An important subclass of the G.S.M. class is the "G.S.M.II" class defined as follows : for such materials the internal variables contain the anelastic strain

$$\beta = (e^{an}, \alpha), \alpha \text{ are other internal state variables}$$

and the free energy $W(e, \beta)$ can be expressed in terms of e^{el} (elastic part of the strain), e^{an} and α as :

$$W(e, \beta) = W(e^{el}, e^{an}, \alpha) .$$

We shall say that the material is a G.S.M.II if its free energy can be split as follows :

$$W(e^{el}, e^{an}, \boldsymbol{\alpha}) = W^{el}(e^{el}) + W^{\alpha}(\boldsymbol{\alpha}) \tag{97}$$

where W^{el} and W^{α} depend only on e^{el} and $\boldsymbol{\alpha}$ respectively. Alternatively a material is a G.S.M.II if W_* can be split into

$$W_*(\boldsymbol{\sigma}, \mathbf{B}) = W_*^{el}(\boldsymbol{\sigma}) + W_*^{\alpha}(\mathbf{A}) \text{ where } \mathbf{A} = -\frac{\partial W}{\partial \boldsymbol{\alpha}} . \tag{98}$$

The normality law (94) for a G.S.M.II becomes

$$\dot{e}^{an} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \mathbf{A}) \qquad \dot{\boldsymbol{\alpha}} = \frac{\partial \Phi}{\partial \mathbf{A}}(\boldsymbol{\sigma}, \mathbf{A}) \tag{99}$$

since it can be proved from (97) that the thermodynamical force associated to e^{an} through (95) is $\boldsymbol{\sigma}$.

Example

The material defined by (72) (73) is a G.S.M. and a G.S.M.II material. Let us set :

$$\text{G.S.M.} \quad \boldsymbol{\beta} = e^{an}, \; W(e, \boldsymbol{\beta}) = \frac{1}{2}\, a(e - \boldsymbol{\beta})(e - \boldsymbol{\beta}), \; \Phi = \phi$$

$$\text{G.S.M.II} \quad \text{no } \boldsymbol{\alpha}, \; W^{el}(e^{el}) = \frac{1}{2}\, a\, e^{el} e^{el}, \; W^{\alpha} = 0, \; \Phi = \phi .$$

The G.S.M.II material is a generalization of (72) (73) in the following sense : we set

$$\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \mathbf{A}), \; \mathbf{E} = (e, 0), \; \mathbf{E}^{el} = (e^{el}, -\boldsymbol{\alpha}), \; \mathbf{E}^{an} = (e^{an}, \boldsymbol{\alpha}) .$$

Then $\mathbf{E} = \mathbf{E}^{el} + \mathbf{E}^{an}$ and (58) is equivalent to :

$$\boldsymbol{\Sigma} = \frac{\partial W}{\partial \mathbf{E}^{el}}(\mathbf{E}^{el}) \qquad \dot{\mathbf{E}}^{an} = \frac{\partial \Phi}{\partial \boldsymbol{\Sigma}}(\boldsymbol{\Sigma}) \tag{100}$$

which generalizes (72) (73).

Let us show that the material defined by (93) is a G.S.M.II material.

Set :

$$\mathbf{A}(x, y) = \boldsymbol{\sigma}^r(x, y)^1 \tag{101}$$

$$W_*^{el}(\boldsymbol{\sigma}^0) = \frac{1}{2} A^{eff}_{pqrs}\, \sigma^0_{rs}\, \sigma^0_{pq} \tag{102}$$

$$W_*^{\alpha}(\boldsymbol{\sigma}^r(y)) = \frac{1}{2|Y|} \int_Y \mathbf{A}(y)\, \boldsymbol{\sigma}^r(y)\, \boldsymbol{\sigma}^r(y)\, dy \tag{103}$$

Since $\boldsymbol{\sigma}^r$ must satisfy

$$\operatorname{div} \boldsymbol{\sigma}^r = 0, \; <\boldsymbol{\sigma}^r> = 0 \quad \boldsymbol{\sigma}^r \cdot \mathbf{n} \text{ antiperiodic}$$

we can show that

$$-\frac{\partial W_*}{\partial \boldsymbol{\sigma}^r}(\boldsymbol{\sigma}^r) = \int_0^t \frac{\partial \phi}{\partial \boldsymbol{\sigma}}(y, \mathbf{C}(y)\, \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^r(y))\, dy \tag{104}$$

which shows that the internal variables associated to the residual stresses are the microscopic anelastic strains. It is then easy to show that, with Φ given by (90), the normality law (99) is equivalent to (93).

Remark 5. – Let us point out a few well known facts on polycristals [20].

a) The macroscopic free energy is the average of the microscopic elastic energy (in fact of the microscopic free energy).

$$\begin{aligned} W_*(\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^r) &= \frac{1}{2|Y|} \int_Y \mathbf{A}(y)(\mathbf{C}(y)\,\boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^r(y))(\mathbf{C}(y)\,\boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^r(y))\, dy \\ &= \left\{ \frac{1}{2|Y|} \int_Y \mathbf{A}(y)\,\mathbf{C}(y)\,\mathbf{C}(y)\, dy \right\} \boldsymbol{\sigma}^0\, \boldsymbol{\sigma}^0 \\ &\quad + \frac{1}{2|Y|} \int_Y \mathbf{A}(y)\,\boldsymbol{\sigma}^r(y)\,\boldsymbol{\sigma}^r(y)\, dy \end{aligned} \tag{105}$$

However *it does not reduce to the macroscopic elastic energy*. According to (105) we observe that in addition to the macroscopic elastic energy (first term of the right-hand side of (105) it contains the elastic energy stored at the microscopic level because of residual stresses (due to anelasticity).

b) *The macroscopic elastic and anelastic strains are not the averages* of the corresponding microscopic quantities. Indeed we have

$$(e^0)^{el} = <\mathbf{C}(y)\, e^{el}(y)> \qquad (e^0)^{an} = <\mathbf{C}(y)\, e^{an}(y)> \tag{106}$$

---

[1] No confusion is possible between the force $\mathbf{A}$ and the elastic compliance $\mathbf{A}(y)$.

c) We notice that an *infinite number* of internal variables is needed to define the homogenized law : at each macroscopic point x we must know the microscopic anelastic strain at every point y belonging to the cell located "under x". A theory with an infinite number of internal variables is rather useless in practice. However it does reflect the complexity of "nature". Trying to restrict the number of internal variables generally leads to approximate theories.

d) The property of G.S.M. demonstrates that in the appropriate space (of generalized stresses and strains) the normality rule holds. It proves that *normality is stable to scale changes,* provided that a high enough number of internal variables is introduced.

e) The macroscopic anisotropy has two origins (refer to (91)). The first one is the shape of the heterogeneities as in the elastic case. The second one is the anisotropy of the residual stresses which enter the dissipative term in (91).

4.3. Another example of homogenization of dissipative materials

We consider an elastic cell which contains a crack, the lips of which are not perfectly lubricated. We can construct a model of microscopic law of contact, similar to the one described by (72) (73) :

$$\boldsymbol{\sigma} = \mathbf{a}\,\mathbf{e}(\mathbf{u}) \quad \text{in } Y - C \quad \text{(C denotes the crack)}, \tag{107}$$

$$\operatorname{div} \boldsymbol{\sigma} = 0 \quad \text{in } Y - C \;, \tag{108}$$

$$[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^- \,, \quad [\mathbf{u}] = [\mathbf{u}]^{el} + [\mathbf{u}]^{an} \quad \text{cf. Fig. 10)}, \tag{109}$$

$$[\mathbf{u}]^{el} = \frac{\partial w_*}{\partial \mathbf{T}} (\boldsymbol{\sigma}.\, \mathbf{n}) \quad (\mathbf{n} \text{ oriented from} - \text{to} +) \tag{110}$$

$$[\dot{\mathbf{u}}]^{an} = \frac{\partial \phi}{\partial \mathbf{T}} (\boldsymbol{\sigma}.\, \mathbf{n}) \tag{111}$$

where w_* and ϕ, which are convex functions of the stress vector $\mathbf{T} = \boldsymbol{\sigma}.\mathbf{n}$, are the elastic energy and the dissipation function of the crack.

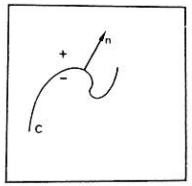

Figure 10. – *Dissipative crack.*

T can be split into its normal and tangential parts :

$$T_n = \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{n} \qquad\qquad \mathbf{T}_t = \boldsymbol{\sigma} \cdot \mathbf{n} - T_n\, \mathbf{n} \,.$$

Using the Legendre transform w of w_*, (110) can be written :

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \frac{\partial w}{\partial \mathbf{v}} ([\mathbf{u}]^{el}), \tag{112}$$

i.e.

$$T_n = \frac{\partial w}{\partial v_n} ([\mathbf{u}]^{el}) \;, \quad \mathbf{T}_t = \frac{\partial w}{\partial \mathbf{v}_t} ([\mathbf{u}]^{el}) \;.$$

This model describes several situations that have been studied in the litterature. Let us mention :

a) Non dissipative crack

$$\text{i)} \quad w(\mathbf{v}) = \begin{cases} 0 \text{ if } v_n \geqslant 0 \\ +\infty \quad \text{otherwise.} \end{cases}$$

This is the frictionless crack with unilateral condition on its lips, studied by Sanchez in [24].

$$\text{ii)} \quad w(\mathbf{v}) = \begin{cases} \frac{1}{2}\, k\, |\mathbf{v}_t|^2 & \text{if } v_n = 0 \\ +\infty \quad \text{otherwise.} \end{cases}$$

This is a model of elastic composites with nonperfectly bonded constituents studied in [23].

b) Dissipative crack

$$\text{i)} \quad w_* = 0 \,, \qquad \phi(\mathbf{T}) = \frac{1}{2\mu} |\mathbf{T}_t|^2 .$$

This is a crack with a linearly viscous friction law for which studies are under way [16].

$$\text{ii) } w(v) = \begin{cases} 0 \text{ if } v_n \geqslant 0 \\ +\infty \quad \text{otherwise} \end{cases} \qquad \phi(T) = \begin{cases} 0 \text{ if } |T_t| \leqslant k \\ +\infty \quad \text{otherwise}. \end{cases}$$

This is a model of crack with unilateral contact on its lips, and a Tresca's law of friction.

$$\text{iii) } w(v) = \begin{cases} 0 \text{ if } v_n \geqslant 0 \\ +\infty \quad \text{otherwise} \end{cases} \qquad \phi(T) = \begin{cases} 0 \text{ if } |T_t| \leqslant k\,|T_n| \\ +\infty \quad \text{otherwise} \end{cases}$$

This is a model of crack with unilateral contact and a *modified standard* Coulomb law of friction. It was established in [25] (the Coulomb law itself is not standard).

We shall present a *formal* way of deriving the homogenized constitutive law of such materials in the framework of G.S.M.. The internal variables are the anelastic part of the jump on the lips of the crack :

$$\boldsymbol{\beta} = [\mathbf{u}]^{an}.$$

We first construct the free energy $W(e^0, \boldsymbol{\beta})$ using the following rule : with e^0 and $\boldsymbol{\beta}$ *given*, we look for the microscopic state $\mathbf{u}$ related to these data and we average the corresponding microscopic free energy :

$$w(e^0, \boldsymbol{\beta}) = \inf_{\substack{u^* \in DP(Y) \\ u^* - e^0 y \text{ periodic}}} \frac{1}{2|Y|} \int_{Y-C} a e(u^*) e(u^*) \, dy + \frac{1}{|Y|} \int_C w([u^*] - \boldsymbol{\beta}) \, ds,$$

and we denote by $\mathbf{u}$ the minimizer of (113). We now identify the forces associated to e^0 and $\boldsymbol{\beta}^1$:

$$\frac{\partial W}{\partial e^0} = \frac{1}{|Y|} \left\{ \int_{Y-C} a e(u) e \left(\frac{\partial u}{\partial e^0}\right) dy + \int_C \frac{\partial w}{\partial v} ([u] - \boldsymbol{\beta}) \frac{\partial [u]}{\partial e^0} \, ds \right\}$$

$$\int_C \frac{\partial W}{\partial \boldsymbol{\beta}} \boldsymbol{\beta}^* \, ds = \frac{1}{|Y|} \left\{ \int_{Y-C} a e(u) e \left(\frac{\partial u}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{\beta}^*\right) dy + \int_C \frac{\partial w}{\partial v} ([u] - \boldsymbol{\beta}) \left(\frac{\partial [u]}{\partial \boldsymbol{\beta}} \boldsymbol{\beta}^* - 1\right) ds \right\}$$

It can easily be shown that the macroscopic strain e^0 and the microscopic displacement $\mathbf{u}$ are related by :

$$e^0_{ij} = \frac{1}{|Y|} \int_{\partial Y} \frac{1}{2} (u_i n_j + u_j n_i) \, ds \tag{114}$$

The variation of Hill's lemma yields

$$\boldsymbol{\tau}^0 e^0 = \frac{1}{|Y|} \left\{ \int_{Y-C} \boldsymbol{\tau}(y) \, e(u(y)) \, dy + \int_C \boldsymbol{\tau} \cdot \mathbf{n} \, [\mathbf{u}] \, ds \right\} \tag{115}$$

with

$$\boldsymbol{\tau}^0 = \frac{1}{|Y|} \int_{Y-C} \boldsymbol{\tau}(y) \, dy$$

Differentiating (67) with respect to e^0 and $\boldsymbol{\sigma}$ we obtain :

$$\boldsymbol{\tau}^0 = \frac{1}{|Y|} \left\{ \int_{Y-C} \boldsymbol{\tau}(y) \, e\left(\frac{\partial \mathbf{u}}{\partial e^0}\right) dy + \int_C \boldsymbol{\tau} \cdot \mathbf{n} \frac{\partial [\mathbf{u}]}{\partial e^0} \, ds \right\} \tag{116}$$

$$0 = \frac{1}{|Y|} \left\{ \int_{Y-C} \boldsymbol{\tau}(y) \, e\left(\frac{\partial \mathbf{u}}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{\beta}^*\right) dy + \int_C \boldsymbol{\tau} \cdot \mathbf{n} \frac{\partial [\mathbf{u}]}{\partial \boldsymbol{\beta}} \boldsymbol{\beta}^* \, ds \right\} \tag{117}$$

These relations allow us to identify the above derivatives

$$\frac{\partial W}{\partial e^0} = \boldsymbol{\sigma}^0, \int_C \frac{\partial w}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{\beta}^* \, ds = \frac{1}{|Y|} \int_C \boldsymbol{\sigma}(y) \cdot \mathbf{n} \, \boldsymbol{\beta}^* \, ds \tag{118}$$

The force associated with e^0 is the macroscopic stress $\boldsymbol{\sigma}^0$, while the force associated with $\boldsymbol{\beta}$ is the stress vector on the lips of the crack. The convexity of W follows from the convexity of w.

The potential Φ is

$$\Phi(B) = \frac{1}{|Y|} \int_C \phi(\boldsymbol{\sigma} \cdot \mathbf{n}) \, ds .$$

The two functions $W(e^0, \boldsymbol{\beta})$ and $\Phi(\mathbf{B})$ define the homogenized material which is a G.S.M. according to § 4.2.

Reduction of the number of internal variables. – The number of internal variables may become finite. It is the case whenever it is possible to show that the microscopic anelastic strain of § 4.1 is everywhere constant (analogue of Taylor's model for polycristals) or piecewise constant. It is also the cas in the model of § 4.3 if the anelastic jumps on the crack are constant, or if the stress vector induced by these jumps is constant. It has been proved by Leguillon-Sanchez [14] that such a situation occurs when a straight crack, parallel to one axis of coordinate of the cell is considered. Their specific problem was slightly different but their arguments apply as such in our context. Let us point out that, since they start from a Coulomb's law of friction for the crack, their homogenized law is not a G.S.M.

Remark 6. – The jump of **u** on the lips of the crack produces an anelastic macroscopic strain :

$$(e^0)^{an}_{ij} = \frac{1}{|Y|} \int_C \frac{1}{2} ([u_i]\, n_j + [u_j]\, n_i)\ ds$$

the elastic strain is

$$(e^0)^{el}_{ij} = \frac{1}{|Y|} \int_{Y-C} e_{ij}(u(y))\ dy$$

and the total strain $\mathbf{e}^0$ given by (114) is the sum of $(\mathbf{e}^0)^{el}$ and $(\mathbf{e}^0)^{an}$.

4.4. Elastic perfectly plastic composites

We now reexamine the situation of § 4.1 in the context of elastic perfectly plastic constituents. For each of them an elasticity convex P (y) is given. It can be proved [27] that the homogenized law is still a G.S.M.II defined as follows. In the space of generalized stresses $\mathbf{\Sigma} = (\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^r(y))$ we define an elasticity convex set by :

$$\mathcal{P} = \{\mathbf{\Sigma} = (\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^r(y)),\ C(y)\,\boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^r(y) \in P(y)\ \forall\, y \in Y\}$$

we also introduce an infinite number of internal variables $\boldsymbol{\alpha}(x, y)$ which are the anelastic microscopic strains. The homogenized law is

$$\left.\begin{array}{l} \mathbf{E}^{el} = ((\mathbf{e}^0)^{el}, -\boldsymbol{\alpha}),\ \mathbf{E}^{el} = \dfrac{\partial W_*}{\partial \mathbf{\Sigma}}(\mathbf{\Sigma}),\ W_* \text{ is given by (105)} \\ \mathbf{\Sigma} \text{ belongs to } \mathcal{P} \\ \dot{\mathbf{E}}^{an} = ((\dot{\mathbf{e}}^0)^{an}, \dot{\boldsymbol{\alpha}}) \text{ is an outer normal vector to } \mathcal{P} \text{ at } \mathbf{\Sigma}. \end{array}\right\} \qquad (119)$$

The law (119) is a G.S.M.II. In the space of generalized stresses and strains the Hill's principle of maximal work is satisfied. This law requires, as for visco-plasticity, an infinite number of internal variables. It is a law of *kinematical hardening* (hardening by the residual stresses). *Nevertheless, the limit convex set of stresses, whose knowledge is required if limit analysis computations are to be performed*, can be easily derived. It is defined as the set of macroscopic stress states $\boldsymbol{\sigma}^0$ for which a plastically admissible microscopic stress state can be found :

$$P^0 = \{\boldsymbol{\sigma}^0 \text{ such that there exists } \boldsymbol{\sigma}(y) \text{ satisfying (120)}\}$$

$$\left.\begin{array}{l} \langle \boldsymbol{\sigma}(y) \rangle = \boldsymbol{\sigma}^0 \\ \operatorname{div} \boldsymbol{\sigma} = 0 \text{ in } Y,\ \boldsymbol{\sigma}\cdot\mathbf{n} \text{ opposite on opposite sides of } Y \\ \boldsymbol{\sigma}(y) \in P(y),\ \forall\, y \in Y \end{array}\right\} \qquad (120)$$

P^0 can be determined through a limit analysis procedure on the basic cell : $\boldsymbol{\sigma}^0$ constitutes the loading. It depends on 6 independent parameters. This limit analysis problem can be solved by standard methods. The boundary conditions are of periodic type, which is always the case in homogenization. There exists a dual approach, the details of which can be found in [27]. No numerical tests have been performed on this method with the exception of [19] which bears some resemblance with (120).

5. Damage and plasticity

Damage and Plasticity are certainly two phenomena which occur together during the loading history of the body. In the preceeding sections we have assumed that they take place at different times : plasticity (often hardening) dominates in the earlier stage, while damage preceedes rupture. However a reasonable theory must account for the combination of both. Even in a purely macroscopic setting, difficulties arise. The first one stems from the necessity of defining precisely the notions of damage and plastic variables.

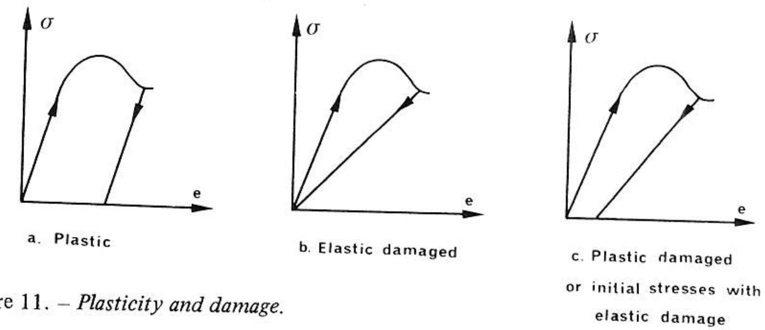

Figure 11. – *Plasticity and damage.*

The classical framework of Plasticity (or G.S.M.) is too broad : if the macroscopic free energy is an arbitrary function $W(e^0, \boldsymbol{\beta})$, $\boldsymbol{\beta}$ can be either a damage variable or a plastic variable :

$$\text{damage} \qquad W(e^0, \boldsymbol{\beta}) = \frac{1}{2}\, a(\boldsymbol{\beta})\, e^0\, e^0$$

$$\text{Plasticity} \qquad W(e^0, \boldsymbol{\beta}) = \frac{1}{2}\, a(e^0 - \boldsymbol{\beta})(e^0 - \boldsymbol{\beta})$$

so that it is impossible to distinguish between the two effects within the G.S.M. class. The correct framework is the damaged G.S.M.II where

$$W(\mathbf{D}, (e^0)^{el}, \boldsymbol{\beta}) = W^{el}(\mathbf{D}, (e^0)^{el}) + W^{\beta}(\mathbf{D}, \boldsymbol{\beta})$$

The internal variables which enter the first term W^{el} are *by definition* the damage variables.
$W^{el}(\mathbf{D}, (e^0)^{el})$ can be derived from the model of §3 by means of homogenization techniques. $W^{\beta}(\mathbf{D}, \boldsymbol{\beta})$ can be determined either by purely macroscopic considerations of the kind exposed in §4, or by homogenization. For a purely macroscopic and thermodynamical study of this coupling we refer to [2]. Let us now briefly investigate the model constructed by homogenization. We start with a combination of the situations of §3 and §4. The free energy is

$$W(\mathbf{D}, (e^0)^{el}, \boldsymbol{\beta}) = \frac{1}{2\,|Y|} \int_{Y - B(D)} a\, e_D^{el}(y)\, e_D^{el}(y)\, dy \tag{121}$$

where $e_D^{el}(y)$ denotes the microscopic elastic strain. Then

$$\mathbf{F}_d = -\frac{\partial W}{\partial \mathbf{D}}(\mathbf{D}, (e^0)^{el}, \boldsymbol{\beta}) = -\frac{1}{|Y|} \int_{Y - B(D)} a\, e_D^{el}(y) \frac{\partial e_D^{el}}{\partial \mathbf{D}}(y)\, dy \tag{122}$$

$$-\frac{1}{2\,|Y|} \int_{\partial B(D)} a\, e_D^{el}(y)\, e_D^{el}(y) \frac{\partial \vec{y}}{\partial D}\, \vec{n}\, ds$$

but it can be shown that it reduces to

$$\mathbf{F}_d = -\frac{1}{2\,|Y|} \int_{\partial B(D)} a\, e_D^{el}(y)\, e_D^{el}(y) \frac{\partial \vec{y}}{\partial \mathbf{D}}\, \vec{n}\, ds \tag{123}$$

The damage force is a "certain" average $\left(\text{depending on } \dfrac{\partial \vec{y}}{\partial \mathbf{D}}\, \vec{n}\right)$ of the *microscopic elastic energy* on the boundary of the microvoids $B(D)$. It is lower than in the elastic case and it vanishes if the medium is rigid plastic. Let us notice that the effective stress which we have earlier defined as the microscopic stress has not an expression as simple as (44) :

$$\boldsymbol{\sigma}(y) = \mathbf{C}^D(y)\, \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^r(y) \tag{124}$$

Thus, for plastic damaged materials the notion of effective stress seems to be difficult to use.

We also notice that, in the case of a material which possesses a microscopic yielding criterion, the theory of §4 gives a macroscopic yielding criterion which depends on $\mathbf{D}$. The rupture of an element can thus have two different causes :

a) rupture by coalescence of the voids

b) rupture by plastic failure if the limit load of the cell is reached for a value of the damage under a prescribed loading.

The coupling of plasticity and damage predicts the rupture of an element for a lower value of the damage parameters.

6. Conclusions

We believe that the homogenization theory is an appropriate tool for deriving macroscopic models from microscopic ones. Anisotropy is taken into account by the theory in a natural way. As it can be seen in this paper the elastic models are now well understood, and they can give a new insight into the theory of damage. The models of plasticity are more delicate to work with : we have shown that properties of normality are preserved in a change of scales provided that we introduce a sufficient number of internal variables. This result is completed by simple models where the number of variables is finite.

The extension to random media, of these results obtained for periodic or quasi-periodic media, is not straightforward. However the mechanisms of the macroscopic laws are the same. Homogenization allows for a better understanding of these mechanisms, while it avoids the use of an overbearing formalism.

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