Convergence of flux-splitting finite volume schemes for hyperbolic scalar conservation laws with a multiplicative stochastic perturbation

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Abstract

We study here explicit flux-splitting finite volume discretizations of multi-dimensional nonlinear scalar conservation laws perturbed by a multiplicative noise with a given initial data in $L^2(\mathbb{R}^d)$. Under a stability condition on the time step, we prove the convergence of the finite volume approximation towards the unique stochastic entropy solution of the equation.

Keywords : Stochastic PDE • first-order hyperbolic equation • Itô integral • multiplicative noise • finite volume method • flux-splitting scheme • Engquist-Osher scheme • Lax-Friedrichs scheme • upwind scheme • Young measures • Kruzhkov smooth entropy.

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1 Introduction

We are interested in the Cauchy problem for a nonlinear hyperbolic scalar conservation law in d space dimensions with a multiplicative stochastic perturbation of type:

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW & \text{in } \Omega\times\mathbb{R}^d\times(0,T), \\ u(\omega,x,0) = u_0(x), & \omega\in\Omega, x\in\mathbb{R}^d, \end{cases} \tag{1}$$

where div is the divergence operator with respect to the space variable (which belongs to $\mathbb{R}^d$), d is a positive integer, $T>0$, $\mathbf{v}\in\mathbb{R}^d$ and $W=\{W_t,\mathcal{F}_t;0\le t\le T\}$ is a standard adapted one-dimensional continuous Brownian motion defined on the classical Wiener space $(\Omega,\mathcal{F},P)$. As mentioned by J.U. Kim [Kim06], by denoting $Q=\mathbb{R}^d\times(0,T)$ this equation has to be understood in the following way: for almost all ω in Ω and for all φ in $\mathcal{D}\big(\mathbb{R}^d\times[0,T)\big)$

$$\begin{aligned} &\int_{\mathbb{R}^d} u_0(x)\varphi(x,0)dx + \int_Q u(\omega,x,t)\partial_t\varphi(x,t) + \mathbf{v}f(u(\omega,x,t)).\nabla_x\varphi(x,t)dxdt \\ = \quad &\int_Q\int_0^t g(u(\omega,x,s))dW(s)\partial_t\varphi(x,t)dxdt. \end{aligned} \tag{2}$$

In order to make the lecture more fluent, we omit the variables ω,x,t and write u instead of $u(\omega,x,t)$.

Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. The mathematical stake consists in introducing a selective criterion in order to identify the physical solution. In the present work we consider a stochastic version of the entropy condition proposed by S.N. Kruzhkov in the 70s, the one used in [BVW12] and presented in Section 2.
We assume the following hypotheses:

H_1: $u_0\in L^2(\mathbb{R}^d)$.

H_2: $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz-continuous function with $f(0)=0$.

H_3: $g:\mathbb{R}\to\mathbb{R}$ is a Lipschitz-continuous function with $g(0)=0$.

H_4: g is a bounded function.

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Remark 1 ***(On these assumptions)***

- H_1 *to* H_3 *are used in [BVW12] to prove the well-posedness of Problem (1). Note that, as it is classically done for hyperbolic scalar conservation laws, one can assume by convenience that* $f(0) = 0$ *without loss of generality.*
- $g(0) = 0$ *is a technical condition coming from [BVW12] and is also used in the present work to show a priori estimates on the finite volume approximate solution.*
- H_4 *is probably a technical assumption, it is particularly used p.18 to show the convergence of the term denoted* $S_2^{h,k}$.

1.1 Former results

Only few papers have been devoted to the study of scalar conservation laws with a multiplicative stochastic forcing. Let us mention the work of Feng-Nualart [FN08], where the authors introduced a notion of strong entropy solution in order to prove the uniqueness of the entropy solution. Using vanishing viscosity and compensated compactness arguments, they established the existence of strong entropy solutions. Note that the use of compensated compactness arguments reduced their study to the one-dimensional case and to genuinely nonlinear flux functions. The authors concluded to the uniqueness of stochastic entropy solution by comparing a strong entropy solution with a stochastic entropy solution.

In the recent paper of Chen-Ding-Karlsen [CDK12], the authors proposed a generalization of the work of Feng-Nualart to the multi-dimensional case. They identified a class of nonlinear stochastic balance laws for which uniform spatial BV bound for vanishing viscosity approximations can be achieved. Moreover they established temporal equicontinuity in L^1 of the approximations, uniformly in the viscosity coefficient. They particularly proved that this stochastic problem is well-posed by using a uniform spatial BV-bound.

Using a kinetic formulation, Debussche-Vovelle [DV10] proved the first complete well-posedness result for multi-dimensional scalar conservation laws set in a d-dimensional torus and driven by a general multiplicative noise. As an extension of this work, in a recent paper Hofmanová [Hof14] presents a Bhatnagar-Gross-Krook-like approximation of this problem. Using the stochastic characteristics method the author establishes the existence of an approximate solution and show its convergence to the kinetic solution of [DV10].

Under assumptions H_1 to H_3 and by the way of Young measure-valued solutions, Bauzet-Vallet-Wittbold [BVW12] proved a result of existence and uniqueness of the solution to the multi-dimensional Cauchy problem in $L^2(\Omega \times Q)$. Since the method consists in comparing a weak measure-valued entropy solution to a regular one (the viscous solution in this case) and not to a strong one, the authors could consider very general assumptions on the data. In the present work, we will use their theoretical results.

In Bauzet-Vallet-Wittbold [BVW14] the authors investigated the Dirichlet Problem for equation (1) set in a bounded domain D of $\mathbb{R}^d$ with homogeneous boundary conditions. They proved a result of existence and uniqueness of the stochastic entropy solution by using the concept of measure-valued solutions and Kruzhkov semi-entropy formulations.

Concerning the study of numerical experiments for scalar conservation laws with multiplicative noise, there is also, to our knowledge, few papers and none of them proposes a convergence study for a space and time discretization of the problem. Let us cite the work of Holden-Risebro [HR91] where a time-discretization of the equation is proposed by the use of an operator-splitting method. They proposed a result of pathwise convergence to prove the existence of pathwise weak solutions to the Cauchy problem for (1) in the one-dimensional case.

In the recent paper of Bauzet [Bau14], a generalization of the work of Holden-Risebro [HR91] is proposed in a bounded domain D of $\mathbb{R}^d$. The author proved that the pathwise weak solution obtained in [HR91] is the unique entropy weak solution of the stochastic conservation law and that the whole sequence of approximation given by the time-splitting scheme converges in $L^p(\Omega \times Q)$ for any finite p. As previously, the convergence study only concerns a time-discretization of the equation. Note that the main result of such a paper is obtained by using the theoretical study of Bauzet-Vallet-Wittbold [BVW14].

Let us mention the paper of Kröker [Krö08] where the author studied well-posedness of a scalar conservation law perturbed by an additive random noise term. In a first part, they proposed a full time-space finite volume method in one and two spatial dimensions but without any convergence study. In a second part, numerical experiments are realized on a few model problems. Note that the stochastic (Itô) part of the equation is approximated by the Euler-Maruyama method.

In the recent work of Kröker-Rohde [KR12] the authors were interested in a method of handling the finite volume schemes for the approximate solution of the Cauchy problem for a hyperbolic balance law with random noise and investigated on a space-discretization of the equation. For a class of strongly monotone numerical fluxes they established the pathwise convergence of a semi-discrete finite volume solution towards a stochastic entropy solution. The main tool was a stochastic version of the compensated compactness approach. It avoids the use of a maximum principle and total-variation estimates but restricts the study to the one-dimensional case and to the use of genuinely nonlinear flux functions.

1.2 Goal of the study and outline of the paper

The aim of this paper is to fill the gap left by the previous authors by introducing a convergence result for a both space and time discretization of multi-dimensional nonlinear scalar conservation laws forced by a multiplicative noise. More precisely, under assumptions H_1 to H_4, we introduce a flux-splitting finite volume scheme for the discretization of Problem (1) and show that the finite volume approximate solution converges in $L^p(\Omega \times Q)$ for all $1 \leqslant p < 2$ to the unique stochastic entropy solution of the equation.
Note that the main difficulty of this study is to choose suitable tools of the finite volume framework compatible with the stochastic one and the restrictions brought by the noise. As we will see thereafter, there is essentially three main constraints to keep in mind:

- Firstly, the use of classical Kruzhkov's entropies seems difficult for the discrete entropy inequalities since the stochastic version of the entropy formulation contains a new term involving the second order derivative of the entropy (see Definition 1). Although this new term is nonnegative, it is unfortunately not in the good side of the inequality and can't be removed of the formulation. In this way, passing to the limit as in the deterministic case to get a formulation with Kruzhkov's entropies is not possible here. This point restricts the available technics of the deterministic finite volume framework to the one involving smooth entropies. Hence we followed some ideas of the paper of Champier-Gallouët-Herbin [CGH93] and adapted them to the stochastic case to show the convergence of the method. In such a paper, the authors were interested in the discretization of a nonlinear hyperbolic equation and proved the convergence of the solution given by an upwind finite volume scheme towards the unique entropy weak solution of their problem using smooth entropies.
- Secondly, due to the construction of the Itô integral, an explicit discretization of the noise term seems to be a more natural choice than an implicit one, see Remark 7.
- Thirdly, note that since the increments of the Brownian motion are not L^∞_ω, even if $u_0 \in L^\infty(\mathbb{R}^d)$ a $L^\infty_{\omega,x,t}$ bound for the finite volume approximate solution is not possible, see Remark 10.

The paper is organized as follows. In Section 2, we recall the definition of a stochastic entropy solution for (1) proposed in [BVW12] and the main result of their paper. In Section 3 we define the flux-splitting finite volume scheme used to approximate the stochastic entropy solution of Problem (1). Then, we give the main result of this paper, which states the convergence of the approximate solution towards the unique stochastic entropy solution of the equation. We also give a few examples of classical flux-splitting finite volume schemes. The remainder of the paper is devoted to the proof of this convergence result. For the sake of readability, the proof will be established only in the case of a nondecreasing flux function f, which leads to an upwind finite volume scheme. But the extension to the case of a general flux-splitting scheme is straightforward. In Section 4, we present firstly the upwind finite volume scheme used to approximate the solution of our problem. In a second time, several preliminary results satisfied by the finite volume approximate solution denoted $u_{\mathcal{T},k}$ are stated. Then in Section 5 we present a result of convergence of $u_{\mathcal{T},k}$ towards the unique stochastic entropy solution of Problem (1).

1.3 Notations

First of all, we need to introduce some notations and make precise the functional setting.

- $Q = \mathbb{R}^d \times (0,T)$.
- Throughout the paper, we denote by C_f and C_g the Lipschitz constants of f and g.
- $|x|$ denotes the euclidian norm of x in $\mathbb{R}^d$ and $x.y$ the usual scalar product of x and y in $\mathbb{R}^d$.
- $||.||_\infty$ denotes the $L^\infty(\mathbb{R})$ norm.
- $V = |\mathbf{v}| \in \mathbb{R}$ denotes the euclidian norm of $\mathbf{v}$ in $\mathbb{R}^d$.
- $E[.]$ denotes the expectation, *i.e.* the integral over Ω with respect to the probability measure P.
- $\mathcal{D}^+\left(\mathbb{R}^d \times [0,T)\right)$ denotes the subset of nonnegative elements of $\mathcal{D}(\mathbb{R}^d \times [0,T))$.
- For a given separable Banach space X we denote by $\mathcal{N}^2_w(0,T,X)$ the space of the predictable X-valued processes (cf. Da Prato-Zabczyk [DPZ92] p.94 for example). This space is the space $L^2\big((0,T) \times \Omega, X\big)$ for the product measure $dt \otimes dP$ on $\mathcal{P}_T$, the predictable σ-field (*i.e.* the σ-field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $(s,t] \times A$ for any $A \in \mathcal{F}_s$).
  If $X = L^2(\mathbb{R}^d)$, one gets that $\mathcal{N}^2_w(0,T,L^2(\mathbb{R}^d)) \subset L^2(\Omega \times Q)$.
- $\mathcal{A}$ the set of any $C^2(\mathbb{R})$ convex functions such that the support of η'' is compact. Note that it implies in particular that η'' and η' are bounded functions.
- F^η denotes the entropy flux defined for any $a \in \mathbb{R}$ and for any smooth function $\eta \in \mathcal{A}$ by
  $F^\eta(a) = \displaystyle\int_0^a \eta'(\sigma) f'(\sigma) d\sigma$. Note in particular that F^η is a Lipschitz-continuous function.

2 The continuous problem

Let us recall the definitions and the result introduced in the paper of BAUZET-VALLET-WITTBOLD [BVW12]. These results are obtained under hypotheses H_1 to H_3.

Definition 1 ***(Stochastic entropy solution)***
A function u of $\mathcal{N}^2_w\big(0,T,L^2(\mathbb{R}^d)\big)\cap L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d)\big)$ is an entropy solution of the stochastic scalar conservation law (1) with the initial condition $u_0\in L^2(\mathbb{R}^d)$, if P-a.s in Ω, for any $\eta\in\mathcal{A}$ and for any $\varphi\in\mathcal{D}^+\big(\mathbb{R}^d\times[0,T)\big)$

$$\begin{aligned}0 \;\leqslant\; & \int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)dx+\int_Q\eta(u)\partial_t\varphi(x,t)dxdt+\int_Q F^\eta(u)\mathbf{v}.\nabla_x\varphi(x,t)dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}\eta'(u)g(u)\varphi(x,t)dxdW(t)+\frac{1}{2}\int_Q g^2(u)\eta''(u)\varphi(x,t)dxdt.\end{aligned}$$

Remark 2 *As mentioned in Section 1.2, the stochastic version of the entropy inequalities limits ourselves to the use of smooth entropies. Indeed, one is not able to get a formulation with Kruzhkov's entropies due to the second order term $\int_Q g^2(u)\eta''(u)\varphi dxdt$.*

Remark 3 *Any entropy solution in the sense of Definition 1 is a weak solution, i.e. it satisfies the weak formulation* (2) *(See [BVW12] Remark 2.6 p.669).*

For technical reasons, as in [BVW12], we also need to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the convergence of the approximate solution to a measure-valued entropy solution. Then, thanks to the result of uniqueness stated in Theorem 1, we will be able to deduce the convergence of the approximate solution to the unique stochastic entropy solution of (1).

Definition 2 ***(Measure-valued entropy solution)***
A function $\mathbf{u}$ of $\mathcal{N}^2_w\big(0,T,L^2(\mathbb{R}^d\times(0,1))\big)\cap L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d\times(0,1))\big)$ is a measure-valued entropy solution of the stochastic scalar conservation law (1) with the initial condition $u_0\in L^2(\mathbb{R}^d)$, if P-a.s in Ω, for any $\eta\in\mathcal{A}$ and for any $\varphi\in\mathcal{D}^+\big(\mathbb{R}^d\times[0,T)\big)$

$$\begin{aligned}0 \;\leqslant\; & \int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)dx+\int_Q\int_0^1\eta(\mathbf{u}(.,\alpha))\partial_t\varphi(x,t)d\alpha dxdt+\int_Q\int_0^1 F^\eta(\mathbf{u}(.,\alpha))\mathbf{v}.\nabla_x\varphi(x,t)d\alpha dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}\int_0^1\eta'(\mathbf{u}(.,\alpha))g(\mathbf{u}(.,\alpha))\varphi(x,t)d\alpha dxdW(t)+\frac{1}{2}\int_Q\int_0^1 g^2(\mathbf{u}(.,\alpha))\eta''(\mathbf{u}(.,\alpha))\varphi(x,t)d\alpha dxdt.\end{aligned}$$

And the main result of [BVW12] is

Theorem 1 *Under assumptions H_1 to H_3 there exists a unique measure-valued entropy solution for the Problem (1) and this solution is obtained by viscous approximation. Moreover, it is the unique stochastic entropy solution in the sense of Definition 1.*

Remark 4 *The unique stochastic entropy solution of Problem (1) given by Theorem 1 satisfies the initial condition in the following sense: for any compact set $K\subset\mathbb{R}^d$*

$$\operatorname*{ess\,lim}_{t\to0^+} E\left[\int_K |u(\omega,x,t)-u_0(x)|dx\right]=0,$$

see [BVW12] Remark 2.7 p.670.

Remark 5 *Following* VALLET *[Val08] Section 6.1, if we assume in addition the following hypotheses*

(i) $0\leqslant u_0(x)\leqslant 1$ *for almost all x in $\mathbb{R}^d$.*

(ii) *suppg* $\subset[0,1]$.

then we can show that $0\le u\le 1$. Indeed, thanks to the Itô formula, this maximum principle is direct for the viscous solution u_ϵ, then it is conserved at the limit for u.

3 Main result

In the sequel, assume that assumptions H_1 to H_4 hold. Let us first give a definition of the admissible meshes for the finite volume scheme.

3.1 Meshes and scheme

Definition 3 ***(Admissible mesh)*** *An admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ for the discretization of Problem (1) is given by a family of disjoint polygonal connected subset of $\mathbb{R}^d$ such that $\mathbb{R}^d$ is the union of the closure of the elements of $\mathcal{T}$ (which are called control volumes in the following) and such that the common interface of any two control volumes is included in a hyperplane of $\mathbb{R}^d$. It is assumed that $h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty$ and that, for some $\alpha \in \mathbb{R}_+^\star$, we have*

$$\alpha h^d \leq |K|, \quad \text{and} \quad |\partial K| \leq \frac{1}{\alpha} h^{d-1}, \quad \forall K \in \mathcal{T}, \tag{3}$$

where we denote by

- . ∂K *the boundary of the control volume* K.
- . $|K|$ *the d-dimensional Lebesgue measure of* K.
- . $|\partial K|$ *the* $(d-1)$*-dimensional Lebesgue measure of* ∂K.
- . $\mathcal{E}_K$ *the set of interfaces of the control volume* K.
- . $\mathcal{N}(K)$ *the set of control volumes neighbors of the control volume* K.
- . $K|L$ *the common interface between* K *and* L *for any* $L \in \mathcal{N}(K)$.
- . $\mathcal{E}$ *the set of all the interfaces of the mesh* $\mathcal{T}$.
- . $n_{K,\sigma}$ *the unit normal to interface* σ, *outward to the control volume* K, *for any* $\sigma \in \mathcal{E}_K$.

Consider an admissible mesh $\mathcal{T}$ in the sense of Definition 3. In order to compute an approximation of u on $[0,T]$ we take $N \in \mathbb{N}^\star$ and define the time step $k = \frac{T}{N} \in \mathbb{R}_+^\star$. In this way $[0,T] = \bigcup_{n=0}^{N-1} [nk, (n+1)k]$. The discrete unknowns are u_K^n, $n \in \{0, ..., N-1\}$, $K \in \mathcal{T}$. The set $\{u_K^0, k \in \mathcal{T}\}$ is given by the initial condition,

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x)dx, \forall K \in \mathcal{T}. \tag{4}$$

The equations satisfied by the discrete unknowns u_K^n, $n \in \{0, ..., N-1\}$, $K \in \mathcal{T}$, are obtained by discretizing Problem (1). For the discretization of such a problem, we consider the following flux-splitting finite volume scheme: we write f as the sum of a nondecreasing function denoted f_1 and a nonincreasing one denoted f_2 (note that such a decomposition is always possible, since the flux-function f is supposed to be Lipschitz-continuous):
For any $K \in \mathcal{T}$, any $n \in \{0, ..., N-1\}$

$$\begin{aligned} \frac{|K|}{k}(u_K^{n+1} - u_K^n) + \sum_{\substack{\sigma \in \mathcal{E}_K \\ \sigma = K|L}} |\sigma| \Big[(\mathbf{v}.n_{K,\sigma})^+ \big(f_1(u_K^n) + f_2(u_L^n) \big) - (\mathbf{v}.n_{K,\sigma})^- \big(f_1(u_L^n) + f_2(u_K^n) \big) \Big] \\ = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}, \end{aligned} \tag{5}$$

where $W^n := W(nk)$ $\forall n \in \{0, ..., N-1\}$. The approximate finite volume solution $u_{\mathcal{T},k}$ may be defined on $\Omega \times \mathbb{R}^d \times [0,T)$ from the discrete unknowns u_K^n, $K \in \mathcal{T}$, $n \in \{0, ..., N-1\}$ which are computed in (5):

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n \text{ for } \omega \in \Omega, x \in K \text{ and } t \in \big[nk, (n+1)k\big), \tag{6}$$

where $\{u_K^0, K \in \mathcal{T}\}$ is determined by (4).

Remark 6 ***(On the measurability of the approximate finite volume solution)*** *Let us mention that using properties of the Brownian motion, for all* K *in* $\mathcal{T}$ *and all* n *in* $\{0, ..., N-1\}$, u_K^n *is* $\mathcal{F}_{nk}$*-measurable and so, as an elementary process adapted to the filtration* $(\mathcal{F}_t)_{t \geq 0}$, $u_{\mathcal{T},k}$ *is predictable with values in* $L^2(\mathbb{R}^d)$.

Remark 7 ***(On the explicit choice in the stochastic integral)*** *We chose in the present work an explicit discretization for the stochastic term, as it is generally done for the discretization of SDEs and SPDEs. Note that with an implicit discretization of such a term, the scheme may be ill-posed if* g *is nonlinear.*

3.2 Main result

We now state the main result of this paper.

Theorem 2 ***(Convergence to the stochastic entropy solution)*** *Assume that hypotheses H_1 to H_4 hold. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and $k = \frac{T}{N} \in \mathbb{R}_+^\star$ be the time step. Let $u_{\mathcal{T},k}$ be the finite volume approximation defined by (5) and (6). Then $u_{\mathcal{T},k}$ converges to the unique stochastic entropy solution of (1) in the sense of Definition 1, in $L^p_{loc}(\Omega \times Q)$ for any $p < 2$ as h tends to 0 and k/h tends to 0.*

Remark 8 *Under the CFL condition*

$$k \leqslant (1-\xi)\frac{\alpha^2 h}{C_f V} \tag{7}$$

one gets for $\xi = 0$ the $L^\infty_t L^2_{\omega,x}$ stability of $u_{\mathcal{T},k}$ stated in Proposition 1 p.7, and for some $\xi \in (0,1)$ the "weak BV" estimate stated in Proposition 2 p.9. In the deterministic case, condition (7) for some $\xi \in (0,1)$ is sufficient to show the convergence of $u_{\mathcal{T},k}$ to the unique entropy solution of the problem, whereas in the stochastic case this condition doesn't seem to be sufficient to show the convergence of the scheme, that is why we assume the stronger assumption $k/h \to 0$ as $h \to 0$.

Remark 9 *This theorem can easily be generalized to the case of a stochastic finite dimensional perturbation of the form $g(u).dW$ where g takes values into $\mathbb{R}^p$ and W is a p-dimensional Brownian motion.*

Since every Lipschitz-continuous function can be decomposed as the sum of a nondecreasing function and a nonincreasing one, for the sake of readability we will only prove this theorem in the case where the flux f is a nondecreasing Lipschitz-continuous function. In this case, the Scheme (5) leads to an upwind finite volume scheme (see Equations (8)). Note that the extension of the proof to the case of a general Lipschitz-continuous flux function is straightforward. Some preliminary results on the upwind finite volume approximate solution will be established in Section 4 and the proof of Theorem 2 will finally be given in Section 5.

3.3 Examples of flux-splitting finite volume schemes

Here are some classical examples of "flux-splitting schemes" for which the convergence result of the present paper holds:

- The most simple example corresponds to the case where the flux function f is monotone, which leads to an upwind scheme, see Equations (8) below.
- The Engquist-Osher scheme concerns a convex or concave flux-function f. In this case either f is monotone and it comes down to the previous case, or f' vanishes in a unique interval of $\mathbb{R}$. In the second case $\mathbb{R}$ is the union of two intervals and f is monotone on each of them, which leads to a natural splitting.
- The modified Lax-Friedrichs scheme in the sense of [EGH00], whose generalization in the case of an hyperbolic system is called the Rusanov scheme, corresponds to a decomposition of the flux in the following way: $f = f_1 + f_2$ where $f_1(x) = f(x)/2 + Dx$ and $f_2(x) = f(x)/2 - Dx$, with $2D \geqslant C_f$.

4 Preliminary results on the finite volume approximation

Assume in the sequel that the flux function f is additionally nondecreasing. In this way, the flux-splitting scheme (5) is reduced to the following upwind finite volume scheme:
For any $K \in \mathcal{T}$, any $n \in \{0, ..., N-1\}$

$$\begin{cases} \dfrac{|K|}{k}(u_K^{n+1} - u_K^n) + \displaystyle\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{v}.n_{K,\sigma} f(u_\sigma^n) = |K| g(u_K^n)\dfrac{W^{n+1} - W^n}{k}, \\ \qquad\qquad u_K^0 = \dfrac{1}{|K|}\displaystyle\int_K u_0(x)dx, \end{cases} \tag{8}$$

where $W^n := W(nk)\ \forall n \in \{0, ..., N-1\}$ and u_σ^n denotes the upstream value at time nk with respect to σ. More precisely, if σ is the interface between the control volumes K and L, u_σ^n is equal to u_K^n if $\mathbf{v}.n_{K,\sigma} \geq 0$ and to u_L^n if $\mathbf{v}.n_{K,\sigma} < 0$.
The approximate upwind finite volume solution $u_{\mathcal{T},k}$ is as previously defined on $\Omega \times \mathbb{R}^d \times [0,T)$ from the discrete unknowns u_K^n, $K \in \mathcal{T}$, $n \in \{0, ..., N-1\}$ computed in (8):

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n \text{ for } \omega \in \Omega, x \in K \text{ and } t \in \big[nk, (n+1)k\big) \tag{9}$$

4.1 Stability estimates

Let us state several results on the finite volume approximate solution $u_{\mathcal{T},k}$ defined by (8) and (9).

Proposition 1 *($L^\infty_t L^2_{\omega,x}$ **estimate**) Let $T>0$, $u_0 \in L^2(\mathbb{R}^d)$, $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and $k = \frac{T}{N} \in \mathbb{R}_+^\star$ satisfying the Courant-Friedrichs-Levy (CFL) condition*

$$k \leqslant \frac{\alpha^2 h}{C_f V}. \tag{10}$$

Let $u_{\mathcal{T},k}$ be the finite volume approximate solution defined by (8) and (9).
Then we have the following bound

$$\|u_{\tau,k}\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))} \leqslant e^{C_g^2 T/2}\|u_0\|_{L^2(\mathbb{R}^d)}.$$

As a consequence we get

$$\|u_{\mathcal{T},k}\|^2_{L^2(\Omega\times Q)} \leqslant T e^{TC_g^2}\|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

Proof. Let us show by induction on $n\in\{0,..,N-1\}$ the following property:

$$\sum_{K\in\mathcal{T}} |K| E\big[(u_K^n)^2\big] \leqslant (1+kC_g^2)^n \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{P_n}$$

First one has

$$\begin{aligned}\sum_{K\in\mathcal{T}} |K| E\big[(u_K^0)^2\big] &= \sum_{K\in\mathcal{T}} |K| E\left[\left(\frac{1}{|K|}\int_K u_0(x)dx\right)^2\right]\\ &\leqslant \|u_0\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Set $n\in\{0,...,N-1\}$ and assume that (P_n) holds. Since $\operatorname{div}\mathbf{v}=0$, the finite volume scheme (8) can be rewritten as follows:

$$\frac{|K|}{k}(u_K^{n+1}-u_K^n) + \sum_{\sigma\in\mathcal{E}_K} |\sigma| \mathbf{v}.n_{K,\sigma}\big(f(u_\sigma^n) - f(u_K^n)\big) = |K| g(u_K^n)\frac{W^{n+1}-W^n}{k},$$

and therefore, using the definition of u_σ^n it is equivalent to

$$\frac{|K|}{k}(u_K^{n+1}-u_K^n) + \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma| (\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n) - f(u_L^n)\Big) = |K| g(u_K^n)\frac{W^{n+1}-W^n}{k}.$$

Let us multiply this scheme by u_K^n. We get, by using formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1}-u_K^n$ and $b = u_K^n$,

$$\begin{aligned}
\frac{|K|}{k}[u_K^{n+1}-u_K^n]u_K^n &= -\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma| (\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n) - f(u_L^n)\Big)u_K^n\\
&\quad + \frac{|K|}{k} g(u_K^n)(W^{n+1}-W^n)u_K^n\\
\Leftrightarrow\quad \frac{1}{2}\frac{|K|}{k}\big[(u_K^{n+1})^2 - (u_K^n)^2 - (u_K^{n+1}-u_K^n)^2\big] &= -\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma| (\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n) - f(u_L^n)\Big)u_K^n\\
&\quad + \frac{|K|}{k} g(u_K^n)(W^{n+1}-W^n)u_K^n\\
\Leftrightarrow\quad \frac{|K|}{2}\big[(u_K^{n+1})^2 - (u_K^n)^2\big] &= \frac{|K|}{2}(u_K^{n+1}-u_K^n)^2 - k\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma| (\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n) - f(u_L^n)\Big)u_K^n\\
&\quad + |K| g(u_K^n)(W^{n+1}-W^n)u_K^n.
\end{aligned}$$

Using (8) we can replace $(u_K^{n+1}-u_K^n)^2$ and this gives by taking the expectation

$$\frac{|K|}{2}E\left[(u_K^{n+1})^2-(u_K^n)^2\right]=\frac{|K|}{2}E\left[\left(\frac{k}{|K|}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_L^n)-f(u_K^n)\Big)+g(u_K^n)(W^{n+1}-W^n)\right)^2\right]$$
$$-kE\left[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n)-f(u_L^n)\Big)u_K^n\right]+|K|E\Big[g(u_K^n)(W^{n+1}-W^n)u_K^n\Big]$$
$$=\frac{k^2}{2|K|}E\left[\left(\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n)-f(u_L^n)\Big)\right)^2\right]+\frac{k|K|}{2}E\left[(g(u_K^n))^2\right]$$
$$-kE\left[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n)-f(u_L^n)\Big)u_K^n\right].$$

Using Cauchy-Schwarz inequality and Assumptions (3) on the mesh we get

$$\frac{|K|}{2}E\left[(u_K^{n+1})^2-(u_K^n)^2\right]\le\frac{k^2}{2|K|}E\left[\left(\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_L^n)-f(u_K^n)\Big)^2\right)\left(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\right)\right]$$
$$-kE\left[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_K^n)-f(u_L^n)\Big)u_K^n\right]+\frac{k|K|}{2}E\left[(g(u_K^n))^2\right]$$
$$\le kE\left[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\left[\frac{kV}{2\alpha^2h}\Big(f(u_L^n)-f(u_K^n)\Big)^2-\Big(f(u_K^n)-f(u_L^n)\Big)u_K^n\right]\right]$$
$$+\frac{k|K|}{2}E\left[(g(u_K^n))^2\right]$$

where we have used that

$$\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\le V|\partial K|\le\frac{V|K|}{\alpha^2h}.$$

Moreover, consider the function $\Phi(a)=\int_0^a sf'(s)ds$ defined for any $a\in\mathbb{R}$ and note that $0\leqslant\Phi(a)\leqslant C_f a^2$. Using the technical Lemma 4.5 p.107 in [EGH00] on monotone functions which states that for any $a,b\in\mathbb{R}$ we have

$$\left|\int_a^b f(s)-f(a)ds\right|\geqslant\frac{1}{2C_f}[f(b)-f(a)]^2,$$

and therefore since f is supposed to be nondecreasing

$$b\Big(f(b)-f(a)\Big)\geqslant\frac{1}{2C_f}[f(b)-f(a)]^2+\Phi(b)-\Phi(a).$$

one shows that (thanks to the CFL Condition (10))

$$\Phi(u_L^n)-\Phi(u_K^n)+\left[f(u_K^n)-f(u_L^n)\right]u_K^n-\frac{kV}{2\alpha^2h}\left[f(u_K^n)-f(u_L^n)\right]^2\geqslant0.$$

In this way

$$\frac{|K|}{2}E\left[(u_K^{n+1})^2-(u_K^n)^2\right]\le\frac{k|K|}{2}C_g^2E\left[(u_K^n)^2\right]+k\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\Phi(u_L^n)-\Phi(u_K^n)\Big].$$

Note that

$$\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\Phi(u_L^n)-\Phi(u_K^n)\Big]=\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|E\Big[(\mathbf{v}.n_{L,\sigma})^+\Phi(u_L^n)-(\mathbf{v}.n_{K,\sigma})^-\Phi(u_K^n)\Big].$$

By summing on each control volume K and using the fact that $\operatorname{div}\mathbf{v}=0$ we obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\Phi(u_L^n)-\Phi(u_K^n)\Big] &= \sum_{\substack{\sigma\in\mathcal{E}\\ \sigma=K|L\\ \mathbf{v}.n_{K,\sigma}\leq 0}} -|\sigma|\mathbf{v}.n_{K,\sigma}E[\Phi(u_L^n)]+|\sigma|\mathbf{v}.n_{K,\sigma}E[\Phi(u_K^n)]\\ &= \sum_{\substack{\sigma\in\mathcal{E}\\ \sigma=K|L\\ \mathbf{v}.n_{K,\sigma}\leq 0}} |\sigma|\mathbf{v}.n_{L,\sigma}E[\Phi(u_L^n)]+|\sigma|\mathbf{v}.n_{K,\sigma}E[\Phi(u_K^n)]\\ &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})E[\Phi(u_K^n)]\\ &= \sum_{K\in\mathcal{T}}E[\Phi(u_K^n)]\int_K \operatorname{div}(\mathbf{v})dx\\ &= 0.\end{aligned}$$

In this way, using (P_n) we get

$$\begin{aligned}\sum_{K\in\mathcal{T}}|K|E\Big[\big(u_K^{n+1}\big)^2\Big] &\leqslant \sum_{K\in\mathcal{T}}|K|(1+kC_g^2)E\big[(u_K^n)^2\big]\\ &\leqslant (1+kC_g^2)^{n+1}\|u_0\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

We deduce that (P_{n+1}) holds, and we conclude by induction that

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}\leqslant e^{C_g^2T/2}\|u_0\|_{L^2(\mathbb{R}^d)}.$$

This gives the $L_t^\infty L^2_{\omega,x}$ stability of the approximate solution. As a consequence, we have

$$\begin{aligned}\|u_{\mathcal{T},k}\|^2_{L^2(\Omega\times Q)} &= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}k|K|E\big[(u_K^n)^2\big]\\ &\leqslant Te^{C_g^2T}\|u_0\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

■

Remark 10 ***(On a $L^\infty_{\omega,x,t}$ estimate)*** *As mention in Remark 5, if one is concerned by the modeling of fluid flow in porous media, if the stochastic entropy solution u has to be a saturation, one gets that $0\leqslant u\leqslant 1$ as soon as hypotheses (i) and (ii) (of Remark 5) are fulfilled. Note that if we assume that $u_0\in L^\infty(\mathbb{R}^d)$ with $0\leq u_0\leq 1$, this bound does not hold for the approximate solution $u_{\mathcal{T},k}$, and this approximation is even unbounded in $L^\infty(\Omega\times Q)$. This is due to the fact that the increments of the Brownian motion are not bounded.*
For example let $\{u_K^n, K\in\mathcal{T}, n\in\{0,...,N-1\}\}$ be given by the finite volume scheme (8) with d=1, $\mathbf{v}=1$, $f(x)=x$, $g(x)=x(1-x)1_{[0,1]}(x)$, $0<\epsilon<1$ and assume that for all $K\in\mathcal{T}$ and all $n\in\{0,...,N-1\}$, $u_K^0=1-\epsilon$. Then

$$u_K^1=1-\epsilon+g(1-\epsilon)W^1.$$

Denote by $a=g(1-\epsilon)>0$. Since $W^1\sim\mathcal{N}(0,k)$, $\mathbb{P}(W^1>\frac{\epsilon}{a})>0$ and so $\mathbb{P}(u_K^1>1)>0$. Indeed, we can even prove that u_K^1 does not belong to $L^\infty(\Omega)$.

4.2 Weak BV estimate

Proposition 2 ***(Weak BV estimate)*** *Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $T>0$, $N\in\mathbb{N}^\star$ and let $k=\frac{T}{N}\in\mathbb{R}_+^\star$ satisfying the CFL condition*

$$k\leqslant\frac{(1-\xi)\alpha^2h}{C_fV},\tag{11}$$

for some $\xi\in(0,1)$.
Let $\{u_K^n, K\in\mathcal{T}, n\in\{0,...,N-1\}\}$ be given by the finite volume scheme (8).
Then there exists $C_1\in\mathbb{R}_+^\star$, only depending on T,u_0,ξ,C_f and C_g such that

$$\sum_{n=0}^{N-1}k\sum_{\substack{\sigma\in\mathcal{E}\\ \sigma=K|L}}|\sigma|\,|\mathbf{v}.n_{K,\sigma}|E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big]\leqslant C_1.$$

Let $T>0$ and $R>0$ be such that $h<R$, we take $N\in\mathbb{N}^\star$ and define $k=\frac{T}{N}\in\mathbb{R}_+^\star$. We also define $\mathcal{T}_R=\{K\in\mathcal{T}$ such that $K\subset B(0,R)\}$. Then there exists $C\in\mathbb{R}_+^\star$, only depending on R,d,T,α,u_0,ξ,C_f and C_g such that

$$\sum_{n=0}^{N-1}k\sum_{\substack{\sigma\in\mathcal{E}^R\\ \sigma=K|L}}|\sigma|\,|\mathbf{v}.n_{K,\sigma}|E\Big[\big|f(u_K^n)-f(u_L^n)\big|\Big]\leqslant Ch^{-1/2},$$

where $\mathcal{E}^R$ denotes the set of interfaces of $\mathcal{T}_R$.

Proof. Multiplying the first equation of (8) by ku_K^n, taking the expectation and summing over $K\in\mathcal{T}$ and $n=0,...,N-1$ yields $A+B=C$ with

$$A=\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|E\big[(u_K^{n+1}-u_K^n)u_K^n\big]$$

$$B=\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}k|\sigma|(\mathbf{v}.n_{K,\sigma})\sum_{\sigma\in\mathcal{E}_K}E\big[f(u_\sigma^n)u_K^n\big]$$

$$=\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}k|\sigma|(\mathbf{v}.n_{K,\sigma})^-\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}E\big[\big(f(u_K^n)-f(u_L^n)\big)u_K^n\big]$$

$$C=\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|E\big[g(u_K^n)u_K^n(W^{n+1}-W^n)\big]=0.$$

Note that the term C is equal to 0 since $g(u_K^n)u_K^n$ is $\mathcal{F}_{nk}$-measurable, it is therefore independent of the increment $W^{n+1}-W^n$.
Using the formula $ab=\frac{1}{2}[(a+b)^2-a^2-b^2]$ with $a=u_K^{n+1}-u_K^n$ and $b=u_K^n$ we get

$$A=-\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|E\left[\left(u_K^{n+1}-u_K^n\right)^2\right]+\frac{1}{2}\sum_{K\in\mathcal{T}}|K|E[(u_K^N)^2-(u_K^0)^2],$$

and using the Scheme (8) gives

$$\begin{aligned}A &= -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|E\left[\left(g(u_K^n)(W^{n+1}-W^n)-\frac{k}{|K|}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_K^n)-f(u_L^n)\big)\right)^2\right]\\ &\quad+\frac{1}{2}\sum_{K\in\mathcal{T}}|K|E[(u_K^N)^2-(u_K^0)^2]\\ &:= A_1+A_2.\end{aligned}$$

Since $2\dfrac{k}{|K|}g(u_K^n)\big(f(u_K^n)-f(u_L^n)\big)$ is $\mathcal{F}_{nk}$-measurable it is therefore independent of the increment $W^{n+1}-W^n$, so that

$$\begin{aligned}A_1 &= -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\left\{k|K|E\Big[\big(g(u_K^n)\big)^2\Big]+\frac{k^2}{|K|}E\left[\left(\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_K^n)-f(u_L^n)\big)\right)^2\right]\right\}\\ &\geqslant -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\left\{k|K|E\Big[\big(g(u_K^n)\big)^2\Big]+(1-\xi)\frac{k}{C_f}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big]\right\},\end{aligned}$$

where we have used an argument similar to one used in the proof of Proposition 1 (under the CFL Condition (11)), namely

$$\begin{aligned}\frac{k^2}{|K|}E\left[\left(\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_K^n)-f(u_L^n)\big)\right)^2\right] &\leq \frac{k^2}{|K|}E\left[\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_K^n)-f(u_L^n)\big)^2\right]\left(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\right)\\ &\leq \frac{k^2}{|K|}|\partial K|V\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big]\\ &\leq (1-\xi)\frac{k}{C_f}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big].\end{aligned}$$

In this way, thanks to Proposition 1 there exists $\tilde{C} > 0$ which only depends on T, C_g and $\|u_0\|_{L^2(\mathbb{R}^d)}$ such that

$$\begin{aligned} A &\geqslant -\frac{1}{2C_f}(1-\xi)\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] - \frac{1}{2}\sum_{K\in\mathcal{T}}|K|[u_K^0]^2 - \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|kE\Big[\big(g(u_K^n)\big)^2\Big] \\ &\geqslant -\frac{1}{2C_f}(1-\xi)\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] - \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2}TC_g^2 e^{TC_g^2}\|u_0\|^2_{L^2(\mathbb{R}^d)} \\ &\geqslant -\frac{1}{2C_f}(1-\xi)\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] - \frac{1}{2}\tilde{C}. \end{aligned}$$

We now study the term $B = \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)u_K^n\Big]$. Let us introduce again the function Φ defined on $\mathbb{R}$ by $\Phi(a) = \int_0^a sf'(s)ds$. Using again the technical Lemma 4.5 p.107 in [EGH00], and the same arguments as in the proof of Proposition 1 we get

$$B \geqslant \frac{1}{2C_f}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big]$$

Therefore, since $A + B = C$

$$0 \geqslant \frac{\xi}{2C_f}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] - \frac{1}{2}\tilde{C},$$

which, in turn, gives the existence of $C_1 \in \mathbb{R}_+^\star$, only depending on T, C_f, C_g, ξ and $\|u_0\|_{L^2(\mathbb{R}^d)}$ such that

$$\sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] \leqslant C_1,$$

or equivalently

$$\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}\\ \sigma=K|L}} |\sigma|\,|\mathbf{v}.n_{K,\sigma}| E\Big[\big(f(u_K^n)-f(u_L^n)\big)^2\Big] \leqslant C_1.$$

Set $R > 0$ be such that $h < R$ and define the set $\mathcal{T}_R = \{K \in \mathcal{T}$ such that $K \subset B(0,R)\}$.
Using Cauchy-Schwarz inequality, we finally get

$$\begin{aligned} \left[\sum_{K\in\mathcal{T}_R}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}\sum_{n=0}^{N-1} k|\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big|f(u_K^n)-f(u_L^n)\big|\Big]\right]^2 &\leqslant \left[\sum_{K\in\mathcal{T}_R}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}\sum_{n=0}^{N-1} k|\sigma|(\mathbf{v}.n_{K,\sigma})^- E\Big[\big|f(u_K^n)-f(u_L^n)\big|^2\Big]\right] \\ &\quad \times \left(\sum_{K\in\mathcal{T}_R}\sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}}\sum_{n=0}^{N-1} k|\sigma|(\mathbf{v}.n_{K,\sigma})^-\right) \\ &\leqslant C_1 VT \sum_{K\in\mathcal{T}_R}|\partial K| \\ &\leqslant C_1 VT\frac{1}{\alpha}h^{d-1}\sum_{K\in\mathcal{T}_R} 1 \\ &\leqslant C_1 VT\frac{1}{\alpha}h^{d-1}\frac{|B(0,R)|}{\alpha h^d} \\ &\leqslant C_1 VT\frac{c}{\alpha^2 h}, \end{aligned}$$

for some constant c depending only on $|B(0,R)|$, and thus with $C = (C_1 VT\frac{c}{\alpha^2})^{1/2}$ one gets

$$\sum_{n=0}^{N-1} k \sum_{\substack{\sigma\in\mathcal{E}^R\\ \sigma=K|L}} |\sigma|\,|\mathbf{v}.n_{K,\sigma}| E\Big[\big|f(u_K^n)-f(u_L^n)\big|\Big] \leqslant Ch^{-1/2},$$

which concludes the proof of the lemma. ∎

4.3 Convergence of the finite volume approximate solution

First of all, note that the *a priori* estimates on $u_{\mathcal{T},k}$ only provide (up to a subsequence) weak convergence for $u_{\mathcal{T},k}$. Moreover, due to the nonlinearity of f and g, one needs compactness arguments to pass to the limit in the nonlinear terms and these arguments have to be compatible with the random variable. The concept of Young measures is appropriate here and the technique is based on the notion of narrow convergence of Young measures (or entropy processes), we refer to Balder [Bal00] but also to Eymard-Gallouët-Herbin [EGH95].

Thanks to the *a priori* estimate stated in Proposition 1, the approximate finite volume solution $u_{\mathcal{T},k}$ converges (up to a subsequence still denoted $u_{\mathcal{T},k}$) in the sense of Young measures to an "entropy process" denoted by $\mathbf{u}$ in $L^2\big(\Omega\times Q\times(0,1)\big)$. Precisely, given a Carathéodory function $\Psi:\Omega\times Q\times\mathbb{R}\to\mathbb{R}$ such that $\Psi(.,u_{\mathcal{T},k})$ is uniformly integrable, one has:

$$E\Big[\int_Q \Psi(.,u_{\mathcal{T},k})dxdt\Big]\to E\Big[\int_Q\int_0^1 \Psi(.,\mathbf{u}(.,\alpha))d\alpha dxdt\Big].$$

We recall that a function $\Psi:\Omega\times Q\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function if for almost any $(\omega,x,t)\in\Omega\times Q$ the function $\nu\mapsto\Psi(\omega,x,t,\nu)$ is continuous and for all $\nu\in\mathbb{R}$, the function $(\omega,x,t)\mapsto\Psi(\omega,x,t,\nu)$ is measurable.

Remark 11 ***(On the measurability of*** $\mathbf{u}$***)*** *Since $u_{\mathcal{T},k}$ is bounded in the Hilbert space $\mathcal{N}_w^2\big(0,T,L^2(\mathbb{R}^d)\big)$, by identification one shows that $u_{\mathcal{T},k}\to\int_0^1\mathbf{u}(.,\alpha)d\alpha$ weakly in $L^2(\Omega\times Q)$ so that $\int_0^1\mathbf{u}(.,\alpha)d\alpha$ is a predictable process with values in $L^2(\mathbb{R}^d)$. The interesting point is the measurability of $\mathbf{u}$ with respect to all its variables (ω,x,t,α). Revisiting the work of* Panov *[Pan96] with the σ-field $\mathcal{P}_T\otimes\mathcal{L}(\mathbb{R}^d)$, one shows that $\mathbf{u}$ is measurable for the σ-field $\mathcal{P}_T\otimes\mathcal{L}(\mathbb{R}^d\times]0,1[)$, thus $\mathbf{u}\in\mathcal{N}_w^2\big(0,T,L^2(\mathbb{R}^d\times]0,1[)\big)$. See Appendix A.3.3 p.707 [BVW12].*

Remark 12 ($L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d\times(0,1))\big)$ ***regularity of*** $\mathbf{u}$***)*** *Since the sequence of approximate solution $u_{\mathcal{T},k}$ is bounded in $L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d)\big)$ according to Proposition 1, following [BVW12] Remark 2.4 p.667-668 we show that $\mathbf{u}\in L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}^d\times(0,1))\big)$.*

Note that if one is able to show that $\mathbf{u}$ is a measure-valued entropy solution of Problem (1) in the sense of Definition 2, then, using the reduction result of [BVW12] stated in Theorem 1, we will be able to conclude that all the sequence $u_{\mathcal{T},k}$ converges in $L^1_{\text{loc}}(\Omega\times Q)$ to the unique stochastic entropy solution of (1) in the sense of Definition 1. Since $\mathbf{u}$ satisfied the regularities required by Definition 2, it remains to show that $\mathbf{u}$ satisfies the following entropy inequalities:
$\forall\eta\in\mathcal{A}$, $\forall\varphi\in\mathcal{D}^+(\mathbb{R}^d\times[0,T))$ and P-a.s. in Ω

$$\begin{aligned}0 \leqslant\ & \int_{\mathbb{R}}\eta(u_0)\varphi(x,0)dx+\int_Q\int_0^1\left\{\eta(\mathbf{u}(.,\alpha))\partial_t\varphi(x,t)+F^\eta\big(\mathbf{u}(.,\alpha)\big)\mathbf{v}.\nabla_x\varphi(x,t)\right\}d\alpha dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}\int_0^1\eta'(\mathbf{u}(.,\alpha))g(\mathbf{u}(.,\alpha))\varphi(x,t)d\alpha dxdW(t)\\ &+\frac{1}{2}\int_Q\int_0^1 g^2(\mathbf{u}(.,\alpha))\eta''(\mathbf{u}(.,\alpha))\varphi(x,t)d\alpha dxdt.\end{aligned}$$

This is the aim of the next section.

5 Convergence of the scheme

We propose in this section entropy inequalities satisfied by the finite volume approximate solution and aim to pass to the limit in these formulations in order to show the convergence of the scheme. For technical reason, one needs to consider a time-continuous approximate solution constructed from $u_{\mathcal{T},k}$, denoted $v_{\mathcal{T},k}$ in the sequel.

5.1 A time-continuous approximation

Note that, since $\operatorname{div}\mathbf{v}=0$, the finite volume scheme (8) can be rewritten as :
For any $K\in\mathcal{T}$, any $n\in\{0,...,N-1\}$

$$\begin{cases} u_K^{n+1}=u_K^n+\dfrac{k}{|K|}\displaystyle\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\Big(f(u_\sigma^n)-f(u_K^n)\Big)+g(u_K^n)(W^{n+1}-W^n),\\ u_K^0=\dfrac{1}{|K|}\displaystyle\int_K u_0(x)dx,\end{cases}$$

where $W^n := W(nk)\ \forall n \in \{0, ..., N-1\}$.
Set $K \in \mathcal{T}$, $n \in \{0, ..., N-1\}$ and consider v_K the stochastic process defined on $\Omega \times [nk, (n+1)k]$ from the discrete unknowns u_K^n by :

$$\begin{aligned} v_K(\omega, s) &= u_K^n + \frac{s-nk}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- \Big(f(u_\sigma^n) - f(u_K^n) \Big) + g(u_K^n)(W(s) - W(nk)) \\ &= u_K^n + \int_{nk}^{s} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- \frac{f(u_\sigma^n) - f(u_K^n)}{|K|} dt + \int_{nk}^{s} g(u_K^n) dW(t), \end{aligned} \tag{12}$$

In this way, $v_K(\omega, nk) = u_K^n$ and $v_K(\omega, (n+1)k) = u_K^{n+1}$.
Let us now define the time-continuous approximate solution $v_{\mathcal{T},k}$ on $\Omega \times \mathbb{R}^d \times [0, T)$ by

$$v_{\mathcal{T},k}(\omega, x, t) = v_K(\omega, t), \omega \in \Omega, x \in K \text{ and } t \in [0, T). \tag{13}$$

We now estimate the difference between the continuous approximation $v_{\mathcal{T},k}$ and the finite volume solution $u_{\mathcal{T},k}$.

Proposition 3 *Let $u_0 \in L^2(\mathbb{R}^d)$ and $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and let $k = \frac{T}{N} \in \mathbb{R}_+^\star$ satisfying the CFL Condition (11).*
Let $v_{\mathcal{T},k}$ be the time-continuous approximate solution defined by (12) and (13), and $u_{\mathcal{T},k}$ be the finite volume approximate solution defined by (8) and (9).
Then there exists $C_1, C_2 \in \mathbb{R}_+^\star$ independent of h and k such that

$$||u_{\mathcal{T},k} - v_{\mathcal{T},k}||^2_{L^2(\Omega \times Q)} \leqslant C_1 h + C_2 k.$$

Proof.

$$\begin{aligned} & ||u_{\mathcal{T},k} - v_{\mathcal{T},k}||^2_{L^2(\Omega \times Q)} \\ = & \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \int_K E\left[\left(-g(u_K^n) \big(W(s) - W^n \big) - \frac{s-nk}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- \Big(f(u_\sigma^n) - f(u_K^n) \Big) \right)^2 \right] dx ds \\ = & \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \int_K E\left[\Big(g(u_K^n) \big(W^n - W(s) \big) \Big)^2 \right] + E\left[\Big(\frac{s-nk}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- \big(f(u_\sigma^n) - f(u_K^n) \big) \Big)^2 \right] dx ds \\ \leqslant & \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} |K| k^2 C_g^2 E\left[\big(u_K^n \big)^2 \right] + k \frac{k^2}{|K|} |\partial K| V \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- E\left[\Big(f(u_\sigma^n) - f(u_K^n) \Big)^2 \right] \\ \leqslant & k C_g^2 ||u_{\mathcal{T},k}||^2_{L^2(\Omega \times Q)} + \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} k (1-\xi)^2 \alpha^4 \frac{h^2}{V^2 C_f^2} \frac{1}{\alpha h^d} \frac{h^{d-1}}{\alpha} V \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- E\left[\Big(f(u_\sigma^n) - f(u_K^n) \Big)^2 \right] \\ \leqslant & k C_g^2 ||u_{\mathcal{T},k}||^2_{L^2(\Omega \times Q)} + h \frac{(1-\xi)^2 \alpha^2}{V C_f^2} \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} k \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\mathbf{v}.n_{K,\sigma})^- E\left[\Big(f(u_\sigma^n) - f(u_K^n) \Big)^2 \right] \\ \leqslant & k C_g^2 ||u_{\mathcal{T},k}||^2_{L^2(\Omega \times Q)} + h \frac{(1-\xi)^2 \alpha^2}{V C_f^2} C_1, \end{aligned}$$

where we have used the constant C_1 given by Proposition 2. ■

5.2 Entropy inequalities for the approximate solution

In this section, an entropy estimate of the approximate solution is proved (Proposition 5), which will be used in the proof of convergence of the numerical scheme (Theorem 3). In order to obtain this entropy estimate, some discrete entropy inequalities satisfied by the approximate solution are first derived in the following proposition.

Proposition 4 ***(Discrete entropy inequalities)*** *Assume that hypotheses H_1 to H_4 hold. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and let $k = \frac{T}{N} \in \mathbb{R}_+^\star$ be the time step and assume that*

$$\frac{k}{h} \to 0 \text{ as } h \to 0. \tag{14}$$

Then P-a.s in Ω, for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+\big(\mathbb{R}^d \times [0,T)\big)$:

$$\begin{aligned}
& -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \big(\eta(u_K^{n+1})-\eta(u_K^n)\big)\varphi(x,nk)dx \\
& +\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dxdt \\
& +\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K \eta'(u_K^n)g(u_K^n)\varphi(x,nk)dxdW(t) \\
& +\frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K \eta''(u_K^n)g^2(u_K^n)\varphi(x,nk)dxdt \\
\geqslant\ & R^{h,k}
\end{aligned} \tag{15}$$

where for any P-measurable set A, $E\big[1_A R^{h,k}\big]\to 0$ as $h\to 0$.

Proof. The proof of this proposition will be separate in two steps: in a first time we will show that Inequality (15) holds for a convenient $R^{h,k}$ and in a second time, we will prove that for any P-measurable set A, $E\big[1_A R^{h,k}\big]\to 0$ as $h\to 0$.
Let $T>0$, $u_0\in L^2(\mathbb{R}^d)$, $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N\in\mathbb{N}^\star$ and $k=\frac{T}{N}\in\mathbb{R}_+^\star$. We assume that $k/h\to 0$ as $h\to 0$, in this way we can suppose that the CFL condition

$$k\leqslant \frac{(1-\xi)\alpha^2 h}{C_f V},$$

holds for some $\xi\in(0,1)$. In this manner, the estimates given by Proposition 1 and Proposition 2 hold. Consider $\eta\in\mathcal{A}$ and $\varphi\in\mathcal{D}^+(\mathbb{R}^d\times[0,T))$, thus there exists $R>h$ such that supp$\varphi\subset B(0,R-h)\times[0,T[$. We also define $\mathcal{T}_R=\{K\in\mathcal{T}$ such that $K\subset B(0,R)\}$.

Step 1: Let us show that Inequality (15) holds for a convenient $R^{h,k}$.
The application of Itô's formula to the process v_K defined by Equation (12) and the function $F:(t,v)\in[0,T]\times\mathbb{R}\mapsto\eta(v)\in\mathbb{R}$ on the interval $[nk,(n+1)k]$ yields P-a.s in Ω

$$\begin{aligned}
\eta\big(v_K((n+1)k)\big) = \ & \eta\big(v_K(nk)\big)+\int_{nk}^{(n+1)k}\eta'(v_K(t))\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\frac{f(u_\sigma^n)-f(u_K^n)}{|K|}dt \\
& +\int_{nk}^{(n+1)k}\eta'(v_K(t))g(u_K^n)dW(t) \\
& +\frac{1}{2}\int_{nk}^{(n+1)k}\eta''(v_K(t))g^2(u_K^n)dt.
\end{aligned} \tag{16}$$

Let us multiply Equation (16) by $|K|\varphi_K^n$, where $\varphi_K^n=\dfrac{1}{|K|}\displaystyle\int_K\varphi(x,nk)dx$, and sum for all $K\in\mathcal{T}_R$ and $n\in\{0,...,N-1\}$. One gets P-a.s in Ω

$$\begin{aligned}
\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\big[\eta(u_K^{n+1})-\eta(u_K^n)\big]|K|\varphi_K^n = \ & \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(v_K(t))\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-(f(u_\sigma^n)-f(u_K^n))dt\varphi_K^n \\
& +\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(v_K(t))g(u_K^n)dW(t)|K|\varphi_K^n \\
& +\frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta''(v_K(t))g^2(u_K^n)dt|K|\varphi_K^n.
\end{aligned}$$

This can be written as $A^{h,k}=B^{h,k}+C^{h,k}+D^{h,k}$, where

$$\begin{aligned}
A^{h,k} &= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\big[\eta(u_K^{n+1})-\eta(u_K^n)\big]|K|\varphi_K^n \\
B^{h,k} &= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(v_K(t))\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-(f(u_\sigma^n)-f(u_K^n))dt\varphi_K^n \\
C^{h,k} &= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(v_K(t))g(u_K^n)dW(t)|K|\varphi_K^n \\
D^{h,k} &= \frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta''(v_K(t))g^2(u_K^n)dt|K|\varphi_K^n.
\end{aligned}$$

Let us analyze separately these terms.

1. Study of $A^{h,k}$: we note that $-A^{h,k}$ is equal to the first left hand side term of Inequality (15).

2. Study of $B^{h,k}$: we decompose $B^{h,k}$ in the following way

$$B^{h,k} = B^{h,k} - \tilde{B}^{h,k} + \tilde{B}^{h,k} - B_1^{h,k} + B_1^{h,k},$$

where

$$\tilde{B}^{h,k} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k\eta'(u_K^n) \sum_{\sigma\in\mathcal{E}_K} |\sigma|(\mathbf{v}.n_{K,\sigma})^-(f(u_\sigma^n) - f(u_K^n))\varphi_K^n$$

$$B_1^{h,k} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} |\sigma|(\mathbf{v}.n_{K,\sigma})^-\left[F^\eta(u_\sigma^n) - F^\eta(u_K^n)\right]\varphi_K^n,$$

Firstly, note that

$$\tilde{B}^{h,k} - B_1^{h,k} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} \int_\sigma (\mathbf{v}.n_{K,\sigma})^- \Big[\eta'(u_K^n)\big(f(u_\sigma^n) - f(u_K^n)\big) - \big(F^\eta(u_\sigma^n) - F^\eta(u_K^n)\big)\Big]\varphi_K^n.$$

Since f and η' are nondecreasing one gets

$$\begin{aligned}\eta'(u_K^n)\left[f(u_\sigma^n) - f(u_K^n)\right] - \left[F^\eta(u_\sigma^n) - F^\eta(u_K^n)\right] &= \int_{u_K^n}^{u_\sigma^n} \eta'(u_K^n) f'(s)ds - \int_{u_K^n}^{u_\sigma^n} \eta'(s)f'(s)ds \\ &= \int_{u_K^n}^{u_\sigma^n} \big(\eta'(u_K^n) - \eta'(s)\big)f'(s)ds \\ &\leqslant 0,\end{aligned}$$

thus $\tilde{B}^{h,k} - B_1^{h,k} \leqslant 0$.
Secondly, since $\operatorname{div}\mathbf{v} = 0$, $B_1^{h,k}$ can be rewritten as

$$B_1^{h,k} = -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}F^\eta(u_\sigma^n)\varphi_K^n.$$

By denoting x_σ the center of the edge σ and φ_σ^n the value $\varphi(x_\sigma, nk)$, note that since $n_{K,\sigma} = -n_{L,\sigma}$ if $\sigma = K|L$, and $u_\sigma^n = u_K^n$ if $\mathbf{v}.n_{K,\sigma} \geq 0$ and $u_\sigma^n = u_L^n$ if $\mathbf{v}.n_{K,\sigma} < 0$, one gets

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}F^\eta(u_\sigma^n)\varphi_\sigma^n = 0,$$

and so

$$B_1^{h,k} = -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}F^\eta(u_\sigma^n)\left[\varphi_K^n - \varphi_\sigma^n\right],$$

which can also be rewritten as

$$B_1^{h,k} = -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}F^\eta(u_K^n)\left[\varphi_K^n - \varphi_\sigma^n\right] + R_1^{h,k},$$

where

$$R_1^{h,k} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}\left[F^\eta(u_K^n) - F^\eta(u_\sigma^n)\right]\left[\varphi_K^n - \varphi_\sigma^n\right].$$

Using again the fact that $\operatorname{div}\mathbf{v} = 0$, $B_1^{h,k}$ is also equal to

$$B_1^{h,k} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \left(\sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}\varphi_\sigma^n\right) F^\eta(u_K^n) + R_1^{h,k}.$$

In this manner, $B_1^{h,k}$ can be rewritten as $B_1^{h,k} = T_2^{h,k} + R_1^{h,k}$, where

$$T_2^{h,k} = \sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \mathbf{v}.n_{K,\sigma}\varphi(x,nk)d\gamma(x)F^\eta(u_K^n) + R_2^{h,k},$$

and

$$R_2^{h,k} = \sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} \left[|\sigma|\mathbf{v}.n_{K,\sigma}\varphi_\sigma^n - \int_\sigma \mathbf{v}.n_{K,\sigma}\varphi(x,nk)d\gamma(x)\right]F^\eta(u_K^n).$$

Moreover,

$$k\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \mathbf{v}.n_{K,\sigma}\varphi(x,nk)d\gamma(x)F^\eta(u_K^n) = k\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{\partial K}\mathbf{v}.n_{K,\sigma}\varphi(x,nk)d\gamma(x)F^\eta(u_K^n)$$
$$= k\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \operatorname{div}(\mathbf{v}\varphi(x,nk))\,dxF^\eta(u_K^n)$$
$$= k\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dx,$$

and in this way

$$B_1^{h,k} = R_1^{h,k} + R_2^{h,k} + \sum_{n=0}^{N-1}k\sum_{K\in\mathcal{T}_R}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dx.$$

Finally,

$$\begin{aligned} B^{h,k} &\leqslant B^{h,k} - \tilde{B}^{h,k} + B_1^{h,k} \\ &= B^{h,k} - \tilde{B}^{h,k} + R_1^{h,k} + R_2^{h,k} + \sum_{n=0}^{N-1}k\sum_{K\in\mathcal{T}_R}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dx.\end{aligned}$$

3. Study of $C^{h,k}$: we decompose $C^{h,k}$ in the following way

$$C^{h,k} = C^{h,k} - \tilde{C}^{h,k} + \tilde{C}^{h,k},$$

where

$$\tilde{C}^{h,k} = \sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\int_K\int_{nk}^{(n+1)k}\eta'(u_K^n)g(u_K^n)\varphi(x,nk)dW(t)dx. \tag{17}$$

4. Study of $D^{h,k}$: we decompose $D^{h,k}$ in the following way

$$D^{h,k} = D^{h,k} - \tilde{D}^{h,k} + \tilde{D}^{h,k},$$

where

$$\tilde{D}^{h,k} = \frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K \eta''(u_K^n)g^2(u_K^n)\varphi(x,nk)dxdt. \tag{18}$$

Since P a.s in Ω, $A^{h,k} = B^{h,k} + C^{h,k} + D^{h,k}$, we get

$$\begin{aligned} &-A^{h,k} + \sum_{n=0}^{N-1}k\sum_{K\in\mathcal{T}_R}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dx + \tilde{C}^{h,k} + \tilde{D}^{h,k} \\ \geqslant\ & \tilde{B}^{h,k} - B^{h,k} + \tilde{C}^{h,k} - C^{h,k} + \tilde{D}^{h,k} - D^{h,k} - R_1^{h,k} - R_2^{h,k},\end{aligned}$$

i.e.

$$\begin{aligned} &-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\left(\eta(u_K^{n+1}) - \eta(u_K^n)\right)\varphi(x,nk)dx \\ &+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dxdt \\ &+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K \eta'(u_K^n)g(u_K^n)\varphi(x,nk)dxdW(t) \\ &+\frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K \eta''(u_K^n)g^2(u_K^n)\varphi(x,nk)dxdt \\ \geqslant\ & \tilde{B}^{h,k} - B^{h,k} + \tilde{C}^{h,k} - C^{h,k} + \tilde{D}^{h,k} - D^{h,k} - R_1^{h,k} - R_2^{h,k},\end{aligned}$$

one finally gets Inequality (15) by choosing

$$R^{h,k} = \tilde{B}^{h,k} - B^{h,k} + \tilde{C}^{h,k} - C^{h,k} + \tilde{D}^{h,k} - D^{h,k} - R_1^{h,k} - R_2^{h,k}. \tag{19}$$

Step 2: Let us show that for any P-measurable set A, $E\left[1_A R^{h,k}\right] \to 0$ as $h \to 0$.

Consider A a P-measurable set and let us analyze separately the convergence of $E\Big[1_A\big(\tilde{B}^{h,k}-B^{h,k}\big)\Big]$, $E\Big[1_A\big(\tilde{C}^{h,k}-C^{h,k}\big)\Big]$, $E\Big[1_A\big(\tilde{D}^{h,k}-D^{h,k}\big)\Big]$, $E\Big[1_A\big(R_1^{h,k}\big)\Big]$ and $E\Big[1_A\big(R_2^{h,k}\big)\Big]$.

1. Convergence of $E\Big[1_A\big(\tilde{B}^{h,k}-B^{h,k}\big)\Big]$

Note that here the assumption $k/h\to 0$ as $h\to 0$ is crucial.

$$\begin{aligned}
\Big|E\Big[1_A\big(\tilde{B}^{h,k}-B^{h,k}\big)\Big]\Big| &= \left|E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}1_A\int_{nk}^{(n+1)k}\{\eta'(v_K(s))-\eta'(u_K^n)\}ds\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-(f(u_\sigma^n)-f(u_K^n))\varphi_K^n\right]\right|\\
&= \left|E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}1_A\eta''(\xi_K^n)\{v_K(s)-u_K^n\}ds\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\varphi_K^n\right]\right|\\
&\leqslant \left|E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}1_A\int_{nk}^{(n+1)k}\eta''(\xi_K^n)\frac{s-nk}{|K|}ds\left(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\right)^2\varphi_K^n\right]\right|\\
&\quad+\Bigg|E\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}1_A\int_{nk}^{(n+1)k}\eta''(\xi_K^n)g(u_K^n)\big(W(s)-W(nk)\big)ds\\
&\quad\times\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\varphi_K^n\Big]\Bigg|\\
&=T_1^{h,k}+T_2^{h,k},
\end{aligned}$$

We analyze separately $T_1^{h,k}$ and $T_2^{h,k}$. Note that in both cases, we use the constant $C_1>0$ given by the weak BV estimate of Proposition 2 and the Assumption (3) on the mesh.

$$\begin{aligned}
|T_1^{h,k}| &= \left|E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}1_A\int_{nk}^{(n+1)k}\eta''(\xi_K^n)\frac{s-nk}{|K|}ds\left(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\right)^2\varphi_K^n\right]\right|\\
&\leqslant \|\eta''\|_\infty\|\varphi\|_\infty\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}k\frac{k}{|K|}\left(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\right)\left(\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_\sigma^n)-f(u_K^n)\big)^2\Big]\right)\\
&\leqslant \|\eta''\|_\infty\|\varphi\|_\infty V\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}k\frac{k}{|K|}|\partial K|\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_\sigma^n)-f(u_K^n)\big)^2\Big]\\
&\leqslant \|\eta''\|_\infty\|\varphi\|_\infty V\frac{k}{\alpha^2h}\sum_{n=0}^{N-1}k\sum_{K\in\mathcal{T}_R}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_\sigma^n)-f(u_K^n)\big)^2\Big]\\
&\leqslant \frac{k}{h}\|\eta''\|_\infty\|\varphi\|_\infty\frac{V}{\alpha^2}C_1\\
&\to 0 \text{ as } h\to 0.
\end{aligned}$$

$$\begin{aligned}
|T_2^{h,k}|^2 &= \left|E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}1_A\varphi_K^n\eta''(\xi_K^n)g(u_K^n)\{W(s)-W(nk)\}dxds\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\right]\right|^2\\
&\leqslant E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\big|1_A\eta''(\xi_K^n)\varphi_K^ng(u_K^n)\big|^2ds\right]\\
&\quad\times E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\left|\big[W(s)-W(nk)\big]\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-\big(f(u_\sigma^n)-f(u_K^n)\big)\right|^2ds\right]\\
&\leqslant \frac{kV}{h\alpha^2}\|\eta''\|_\infty^2\|\varphi\|_\infty^2C_g^2\|u_{\mathcal{T},k}\|_{L^2(\Omega\times Q)}^2\sum_{n=0}^{N-1}k\sum_{K\in\mathcal{T}_R}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(\mathbf{v}.n_{K,\sigma})^-E\Big[\big(f(u_\sigma^n)-f(u_K^n)\big)^2\Big]\\
&\leqslant \frac{k}{h}\frac{C_1V}{\alpha^2}\|\eta''\|_\infty^2\|\varphi\|_\infty^2C_g^2Te^{TC_g^2}\|u_0\|_{L^2(\mathbb{R}^d)}^2\\
&\to 0 \text{ as } h\to 0.
\end{aligned}$$

In this way,

$$E\Big[1_A\big(\tilde{B}^{h,k}-B^{h,k}\big)\Big]\to 0 \text{ as } h\to 0.$$

2. Convergence of $E\Big[1_A\big(\tilde{C}^{h,k}-C^{h,k}\big)\Big]$

$$\begin{aligned}\Big|E\Big[1_A\big(\tilde{C}^{h,k}-C^{h,k}\big)\Big]\Big| &= \left|E\left[\sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\int_K 1_A\int_{nk}^{(n+1)k}[\eta'(v_K(t))-\eta'(u_K^n)]g(u_K^n)\varphi(x,nk)dW(t)dx\right]\right|\\ &\leqslant \left|E\left[\sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\int_K 1_A\int_{nk}^{(n+1)k}[\eta'(v_K(t))-\eta'(u_K^n)]g(u_K^n)\left\{\varphi(x,nk)-\varphi(x,t)\right\}dW(t)dx\right]\right|\\ &\quad+\left|E\left[\sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\int_K 1_A\int_{nk}^{(n+1)k}[\eta'(v_K(t))-\eta'(u_K^n)]g(u_K^n)\varphi(x,t)dW(t)dx\right]\right|\\ &= S_1^{h,k}+S_2^{h,k}.\end{aligned}$$

Using successively Cauchy-Schwarz inequality on $\Omega\times B(0,R)$ and Itô isometry one gets

$$\begin{aligned}S_1^{h,k} &= \left|E\left[\sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\int_K 1_A\int_{nk}^{(n+1)k}\big\{\eta'(v_K(t))-\eta'(u_K^n)\big\}g(u_K^n)\big\{\varphi(x,nk)-\varphi(x,t)\big\}dW(t)dx\right]\right|\\ &\leqslant \sqrt{|B(0,R)|}\sum_{n=0}^{N-1}\left[\sum_{K\in\mathcal{T}_R}\int_K E\left[\left(\int_{nk}^{(n+1)k}\big\{\eta'(v_K(t))-\eta'(u_K^n)\big\}g(u_K^n)\big\{\varphi(x,nk)-\varphi(x,t)\big\}dW(t)\right)^2\right]dx\right]^{1/2}\\ &= \sqrt{|B(0,R)|}\sum_{n=0}^{N-1}\left[\sum_{K\in\mathcal{T}_R}\int_K\int_{nk}^{(n+1)k}E\Big[\big\{\eta'(v_K(t))-\eta'(u_K^n)\big\}^2g^2(u_K^n)\big\{\varphi(x,nk)-\varphi(x,t)\big\}^2\Big]dtdx\right]^{1/2}\\ &\leqslant \sqrt{k}\sqrt{|B(0,R)|}2C_g||\varphi_t||_\infty||\eta'||_\infty\sum_{n=0}^{N-1}k\Big(\sum_{K\in\mathcal{T}_R}|K|E\big[(u_K^n)^2\big]\Big)^{1/2}\\ &\leqslant \sqrt{k}\sqrt{|B(0,R)|}2C_g||\varphi_t||_\infty||\eta'||_\infty Te^{TC_g^2/2}||u_0||_{L^2(\mathbb{R}^d)}\\ &\to 0 \text{ as h} \to 0.\end{aligned}$$

Note that here Assumption H_4 on the function g is important:

$$\begin{aligned}\big(S_2^{h,k}\big)^2 &= \left|E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}1_A\int_K\int_{nk}^{(n+1)k}\big\{\eta'(v_K(t))-\eta'(u_K^n)\big\}g(u_K^n)\varphi(x,t)dW(t)dx\right]\right|^2\\ &= \left|E\left[1_A\int_{B(0,R)}\int_0^T\big\{\eta'(v_{\mathcal{T},k})-\eta'(u_{\mathcal{T},k})\big\}g(u_{\mathcal{T},k})\varphi(x,t)dW(t)dx\right]\right|^2\\ &\leqslant |B(0,R)|\int_{B(0,R)}E\left[\left(\int_0^T\big\{\eta'(v_{\mathcal{T},k})-\eta'(u_{\mathcal{T},k})\big\}g(u_{\mathcal{T},k})\varphi(x,t)dW(t)\right)^2\right]dx\\ &= |B(0,R)|\int_{B(0,R)}\int_0^T E\Big[\big\{\eta'(v_{\mathcal{T},k})-\eta'(u_{\mathcal{T},k})\big\}^2g^2(u_{\mathcal{T},k})\varphi^2(x,t)\Big]dtdx\\ &\leqslant |B(0,R)|\|\varphi\|_\infty^2\|\eta''\|_\infty^2\|g\|_\infty^2\|v_{\mathcal{T},k}-u_{\mathcal{T},k}\|_{L^2(\Omega\times Q)}^2\\ &\to 0 \text{ as h}\to 0 \text{ using Proposition 3.}\end{aligned}$$

In this way,

$$E\Big[1_A\big(\tilde{C}^{h,k}-C^{h,k}\big)\Big]\to 0 \text{ as } h\to 0.$$

3. Convergence of $E\Big[1_A\big(\tilde{D}^{h,k}-D^{h,k}\big)\Big]$

$$\begin{aligned}\Big|E\Big[1_A\big(\tilde{D}^{h,k}-D^{h,k}\big)\Big]\Big| &= \left|\frac{1}{2}E\left[\sum_{K\in\mathcal{T}_R}\sum_{n=0}^{N-1}\int_K\int_{nk}^{(n+1)k}1_A\left[\eta''(u_K^n)-\eta''(v_K(t))\right]g^2(u_K^n)\varphi(x,nk)dxdt\right]\right|\\ &\leqslant \frac{1}{2}||g||_\infty^2||\varphi||_\infty||\eta''(u_{\mathcal{T},k})-\eta''(v_{\mathcal{T},k})||_{L^1(\Omega\times B(0,R)\times(0,T))}\\ &\to 0 \text{ as h}\to 0 \text{ using Proposition 3.}\end{aligned}$$

In this way,

$$E\Big[1_A\big(\tilde{D}^{h,k}-D^{h,k}\big)\Big]\to 0 \text{ as } h\to 0.$$

4. Convergence of $E\Big[1_A\big(R_1^{h,k}\big)\Big]$

Thanks to the weak BV estimate stated in Proposition 2, one shows that $E\left[1_A R_1^{h,k}\right] \to 0$ as $h \to 0$. Indeed,

$$\begin{aligned}
\left|E[1_A R_1^{h,k}]\right| &= \left|E\left[\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} 1_A k \sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{v}.n_{K,\sigma}\left[F^\eta(u_K^n) - F^\eta(u_\sigma^n)\right]\left[\varphi_K^n - \varphi_\sigma^n\right]\right]\right| \\
&= \left|E\left[\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} 1_A k \sum_{\sigma\in\mathcal{E}_K} |\sigma|(\mathbf{v}.n_{K,\sigma})^-\left[F^\eta(u_K^n) - F^\eta(u_\sigma^n)\right]\left[\varphi_K^n - \varphi_\sigma^n\right]\right]\right| \\
&\leqslant \|\varphi_x\|_\infty h \sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\left[\left|\int_{u_\sigma^n}^{u_K^n} \eta'(s) f'(s) ds\right|\right] \\
&\leqslant \|\varphi_x\|_\infty \|\eta'\|_\infty h \sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} |\sigma|(\mathbf{v}.n_{K,\sigma})^- E\left[\,|f(u_\sigma^n) - f(u_K^n)|\,\right] \\
&\leqslant \frac{h}{\sqrt{h}} C \|\varphi_x\|_\infty \|\eta'\|_\infty \\
&\to 0 \text{ as } h \to 0,
\end{aligned}$$

where C is the constant given by Proposition 2.

5. Convergence of $E\left[1_A\left(R_2^{h,k}\right)\right]$

By denoting x_σ the center of the edge σ, let us recall that $R_2^{h,k}$ is equal to

$$\sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} \left[|\sigma|\mathbf{v}.n_{K,\sigma}\varphi(x_\sigma, nk) - \int_\sigma \mathbf{v}.n_{K,\sigma}\varphi(x, nk) d\gamma(x)\right] F^\eta(u_K^n).$$

Using the regularity of φ we have for all $x \in \sigma$

$$\varphi(x, nk) = \varphi(x_\sigma, nk) + \varphi'(x_\sigma, nk)(x - x_\sigma) + (x - x_\sigma)\epsilon(x - x_\sigma),$$

where $\epsilon(x - x_\sigma) \to 0$ as $x - x_\sigma \to 0$. In this way,

$$R_2^{h,k} = -\sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} \left[\int_\sigma \mathbf{v}.n_{K,\sigma}\left\{\varphi'(x_\sigma, nk)(x - x_\sigma) + (x - x_\sigma)\epsilon(x - x_\sigma)\right\} d\gamma(x)\right] F^\eta(u_K^n),$$

Since x_σ denotes the center of the edge σ,

$$\int_\sigma \varphi'(x_\sigma, nk)(x - x_\sigma) d\gamma(x) = 0.$$

Moreover,

$$\left|\int_\sigma (x - x_\sigma)\epsilon(x - x_\sigma) d\gamma(x)\right| \leqslant |\sigma| h \bar{\epsilon}(h),$$

where $\bar{\epsilon}(h) \to 0$ as $h \to 0$. Thus,

$$\begin{aligned}
|E[1_A R_2^{h,k}]| &\leqslant V \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \sum_{\sigma\in\mathcal{E}_K} k E\left[|F^\eta(u_K^n)|\right] |\sigma| h \bar{\epsilon}(h) \\
&\leqslant V C_f \|\eta'\|_\infty \bar{\epsilon}(h) \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k \frac{h^d}{\alpha} E\left[|u_K^n|\right] \\
&\leqslant \frac{V}{\alpha^2} C_f \bar{\epsilon}(h) \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} k|K| E\left[|u_K^n|\right] \\
&\leqslant \frac{V}{\alpha^2} C_f \bar{\epsilon}(h) ||u_{\mathcal{T},k}||_{L^1(\Omega\times(0,T)\times B(0,R))} \\
&\to 0 \text{ as } h \to 0.
\end{aligned}$$

To summarize, we proved in this second step that $E\left[1_A R^{h,k}\right] \to 0$ as $h \to 0$ which concludes the proof of the proposition. ■

The following proposition investigates the entropy inequalities which are satisfied by the approximate solution $u_{\mathcal{T},k}$.

Proposition 5 ***(Continuous entropy inequality on the discrete solution)*** *Assume that hypotheses H_1 to H_4 hold. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and let $k = \frac{T}{N} \in \mathbb{R}_+^\star$ be the time step and assume that*

$$\frac{k}{h} \to 0 \text{ as } h \to 0. \tag{20}$$

Then, P-a.s. in Ω, for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+\big(\mathbb{R}^d \times [0,T)\big)$:

$$\begin{aligned} &\int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx + \int_Q \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt + \int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt \\ &+ \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t) + \frac{1}{2}\int_Q \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt \\ \geqslant\ & \tilde{R}^{h,k} \end{aligned} \tag{21}$$

where for any P-measurable set A, $E\big[1_A \tilde{R}^{h,k}\big] \to 0$ as $h \to 0$.

Proof. The proof of this proposition will be separate in two steps: in a first time we will show that Inequality (21) holds for a convenient $\tilde{R}^{h,k}$ and in a second time, we will prove that for any P-measurable set A, $E\big[1_A \tilde{R}^{h,k}\big] \to 0$ as $h \to 0$.
Let $T > 0$, $u_0 \in L^2(\mathbb{R}^d)$, $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and $k = \frac{T}{N} \in \mathbb{R}_+^\star$.We assume that $k/h \to 0$ as $h \to 0$, in this way we can suppose that the CFL condition

$$k \leqslant \frac{(1-\xi)\alpha^2 h}{C_f V},$$

holds for some $\xi \in (0,1)$. In this manner, the estimates given by Proposition 1 and Proposition 2 hold. Consider $\eta \in \mathcal{A}$ and $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0,T))$, thus there exists $R > h$ such that supp$\varphi \subset B(0,R-h) \times [0,T[$. We also define $\mathcal{T}_R = \{K \in \mathcal{T}$ such that $K \subset B(0,R)\}$.

Step 1: Let us show that Inequality (21) holds for a convenient $\tilde{R}^{h,k}$.

Note that the first term of Inequality (15) given by Proposition 4 can be rewritten in the following way:

$$\begin{aligned} &-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\big[\eta(u_K^{n+1})-\eta(u_K^n)\big]\int_K \varphi(x,nk)dx \\ &= \int_k^T \int_{\mathbb{R}^d} \eta(u_{\mathcal{T},k})\varphi_t(x,t-k)dxdt + \sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^0)\varphi(x,0)dx. \end{aligned}$$

Indeed, thanks to the discrete integration by part formula

$$\sum_{n=1}^{N} a_n(b_n - b_{n-1}) = a_N b_N - a_0 b_0 - \sum_{n=0}^{N-1} b_n(a_{n+1}-a_n)$$

and by using the fact that for all x in K $\varphi(x,Nk) = 0$ we get

$$\begin{aligned} \int_k^T \int_{\mathbb{R}^d} \eta(u_{\mathcal{T},k})\varphi_t(x,t-k)dxdt &= \sum_{n=1}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^n)\left[\varphi(x,nk)-\varphi(x,(n-1)k)\right]dx \\ &= -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \big[\eta(u_K^{n+1})-\eta(u_K^n)\big]\varphi(x,nk)dx \\ &\quad + \sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^N)\varphi(x,Nk) - \eta(u_K^0)\varphi(x,0)dx \\ &= -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \big[\eta(u_K^{n+1})-\eta(u_K^n)\big]\varphi(x,nk)dx \\ &\quad - \sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^0)\varphi(x,0)dx. \end{aligned}$$

By denoting

$$\begin{aligned} C_1^{h,k} &= \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t) \\ D_1^{h,k} &= \frac{1}{2}\int_Q \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt \end{aligned}$$

one gets from Inequality (15), Inequality (21) with $\tilde{R}^{h,k}$ defined by

$$\begin{aligned} \tilde{R}^{h,k} = &\ R^{h,k} + \int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)dx - \sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^0)\varphi(x,0)dx \\ &+ \int_Q \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt - \int_k^T \int_{\mathbb{R}^d}\eta(u_{\mathcal{T},k})\varphi_t(x,t-k)dxdt \\ &+ \int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dxdt \\ &+ C_1^{h,k} - \tilde{C}^{h,k} + D_1^{h,k} - \tilde{D}^{h,k}. \end{aligned}$$

where $\tilde{C}^{h,k}, \tilde{D}^{h,k}, R^{h,k}$ are given respectively by (17), (18), (19) in the proof of the previous proposition.

Step 2: Let us show that for any P-measurable set A, $E\big[1_A\tilde{R}^{h,k}\big] \to 0$ as $h \to 0$.
Thanks to Proposition 4, we know that for any P-measurable set A, $E\big[1_A R^{h,k}\big] \to 0$ as $h \to 0$. Then it remains to study the convergence of the following quantities:

$$E\left[1_A\left(\int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)dx - \sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^0)\varphi(x,0)dx\right)\right],$$
$$E\left[1_A\left(\int_Q \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt - \int_k^T\int_{\mathbb{R}^d}\eta(u_{\mathcal{T},k})\varphi_t(x,t-k)dxdt\right)\right],$$
$$E\left[1_A\left(\int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dxdt\right)\right],$$
$$E\Big[1_A\big(C_1^{h,k} - \tilde{C}^{h,k}\big)\Big] \text{ and } E\Big[1_A\big(D_1^{h,k} - \tilde{D}^{h,k}\big)\Big].$$

Let us analyze separately the convergence of these terms as $h \to 0$.

1. Convergence of $E\left[1_A\left(\int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)dx - \sum_{K\in\mathcal{T}_R}\int_K \eta(u_K^0)\varphi(x,0)dx\right)\right]$

Since $u_0 \in L^1_{loc}(\mathbb{R}^d)$, one shows that this term tends to 0 as $h \to 0$.

2. Convergence of $E\left[1_A\left(\int_Q \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt - \int_k^T\int_{\mathbb{R}^d}\eta(u_{\mathcal{T},k})\varphi_t(x,t-k)dxdt\right)\right]$

Using the regularity of the function φ and the *a priori* estimate on $u_{\mathcal{T},k}$, one shows that this term tends to 0 as $h \to 0$.

3. Convergence of $E\left[1_A\left(\int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K F^\eta(u_K^n)\mathbf{v}.\nabla_x\varphi(x,nk)dxdt\right)\right]$

Using again the regularity of the function φ and the *a priori* estimate on $u_{\mathcal{T},k}$, one shows that this term tends to 0 as $h \to 0$.

4. Convergence of $E\Big[1_A\big(C_1^{h,k} - \tilde{C}^{h,k}\big)\Big]$

Using Cauchy-Schwarz inequality on $\Omega \times B(0,R)$ and Itô isometry one gets

$$\begin{aligned}
\left|E\Big[1_A\big(C_1^{h,k} - \tilde{C}^{h,k}\big)\Big]\right| &= \left|E\left[1_A\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\int_{nk}^{(n+1)k}\eta'(u_K^n)g(u_K^n)\{\varphi(x,nk)-\varphi(x,t)\}dW(t)dx\right]\right|\\
&\leqslant \sum_{n=0}^{N-1}\sqrt{|B(0,R)|}\left(\sum_{K\in\mathcal{T}_R}\int_K E\left[\left(\int_{nk}^{(n+1)k}\eta'(u_K^n)g(u_K^n)\{\varphi(x,nk)-\varphi(x,t)\}dW(t)\right)^2\right]dx\right)^{1/2}\\
&= \sum_{n=0}^{N-1}\sqrt{|B(0,R)|}\left(\sum_{K\in\mathcal{T}_R}\int_K\int_{nk}^{(n+1)k}E\left[\left(\eta'(u_K^n)g(u_K^n)\{\varphi(x,nk)-\varphi(x,t)\}\right)^2\right]dtdx\right)^{1/2}\\
&\leqslant \sqrt{k}\sqrt{|B(0,R)|}C_g\|\varphi_t\|_\infty\|\eta'\|_\infty\sum_{n=0}^{N-1}k\Big(\sum_{K\in\mathcal{T}_R}|K|E\big[(u_K^n)^2\big]\Big)^{1/2}\\
&\leqslant \sqrt{k}\sqrt{|B(0,R)|}C_g\|\varphi_t\|_\infty\|\eta'\|_\infty T e^{TC_g^2/2}\|u_0\|_{L^2(\mathbb{R}^d)}\\
&\to 0 \text{ as } h \to 0.
\end{aligned}$$

5. Convergence of $E\Big[1_A\big(D_1^{h,k} - \tilde{D}^{h,k}\big)\Big]$

$$\begin{aligned}
\left|E\Big[1_A\big(D_1^{h,k} - \tilde{D}^{h,k}\big)\Big]\right| &= \left|\frac{1}{2}E\left[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\int_K 1_A\eta''(u_K^n)g^2(u_K^n)\big[\varphi(x,nk)-\varphi(x,t)\big]dxdt\right]\right|\\
&\leqslant \frac{1}{2}k\|\eta''\|_\infty\|g\|_\infty^2\|\varphi_t\|_\infty T|B(0,R)|\\
&\to 0 \text{ as } h \to 0.
\end{aligned}$$

To summarize, we proved in this second step that $E\big[1_A\tilde{R}^{h,k}\big] \to 0$ as $h \to 0$, which concludes the proof of the proposition. ■

5.3 Proof of the convergence

And we prove now the convergence of the finite volume approximation $u_{\mathcal{T},k}$ to the stochastic entropy solution of Problem (1).

Theorem 3 ***(Convergence to the stochastic entropy solution)*** *Assume that hypotheses H_1 to H_4 hold. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and let $k = \frac{T}{N} \in \mathbb{R}_+^\star$ be the time step and assume that*

$$\frac{k}{h} \to 0 \text{ as } h \to 0. \tag{22}$$

Let $u_{\mathcal{T},k}$ be the finite volume approximation defined by (8) and (9). Then $u_{\mathcal{T},k}$ converges in $L^p_{loc}(\Omega \times Q)$ for any $1 \leqslant p < 2$ to the unique stochastic entropy solution of (1) in the sense of Definition 1.

Proof. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3, $N \in \mathbb{N}^\star$ and let $k = \frac{T}{N} \in \mathbb{R}_+^\star$ be the time step such that $k/h \to 0$ as $h \to 0$. In this way we can suppose that the CFL condition

$$k \leqslant \frac{(1-\xi)\alpha^2 h}{C_f V},$$

holds for some $\xi \in (0,1)$. In this manner, the estimates given by Proposition 1 and Proposition 2 hold. Consider A a P-measurable set, $\eta \in \mathcal{A}$, $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0,T))$, thus there exists $R > h$ such that supp$\varphi \subset B(0,R-h) \times [0,T[$. We also define $\mathcal{T}_R = \{K \in \mathcal{T} \text{ such that } K \subset B(0,R)\}$.
Let us multiply Inequality (21) by 1_A and take the expectation. This yields:

$$\begin{aligned}
& E\Big[1_A \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx\Big] + E\Big[1_A \int_Q \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt\Big] + E\Big[1_A \int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt\Big] \\
& + E\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t)\Big] + \frac{1}{2}E\Big[1_A \int_Q \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt\Big] \\
\geqslant\ & E\big[1_A \tilde{R}^{h,k}\big].
\end{aligned} \tag{23}$$

To show the convergence of $u_{\mathcal{T},k}$ towards the unique stochastic entropy solution of our problem, we aim to pass to the limit in the above inequality. Thanks to Proposition 5 we know that for any P-measurable set A, $E\big[1_A \tilde{R}^{h,k}\big] \to 0$ as $h \to 0$. Thus it remains to study the convergence of the left-hand side of (23). Recall that thanks to the *a priori* estimate stated in Proposition 1, $u_{\mathcal{T},k}$ converges (up to a subsequence denoted in the same way) in the sense of Young measures to an “entropy process” denoted by $\mathbf{u}$ in $L^2\big(\Omega \times Q \times (0,1)\big)$ (see Section 4.3).

1. Study of $E\Big[1_A \int_Q \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt\Big]$
Note that $\Psi : (\omega,x,t,\nu) \in \Omega \times Q \times \mathbb{R} \mapsto 1_A(\omega)\eta(\nu)\varphi_t(x,t) \in \mathbb{R}$ is a Carathéodory function such that $\Psi(.,u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q)$, thus

$$E\Big[1_A \int_Q \eta(u_{\mathcal{T},k}(x,t))\varphi_t(x,t)dxdt\Big] \to E\Big[1_A \int_Q \int_0^1 \eta(\mathbf{u}(x,t,\alpha))d\alpha\varphi_t(x,t)dxdt\Big] \text{ as } h \to 0.$$

2. Study of $E\Big[1_A \int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt\Big]$
Since $F^\eta(u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q)$, using the same arguments as previously, we obtain

$$E\Big[1_A \int_Q F^\eta(u_{\mathcal{T},k})\mathbf{v}.\nabla_x\varphi(x,t)dxdt\Big] \to E\Big[1_A \int_Q \int_0^1 F^\eta(\mathbf{u}(x,t,\alpha))\mathbf{v}.\nabla_x\varphi(x,t)d\alpha dxdt\Big] \text{ as } h \to 0.$$

3. Study of $E\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t)\Big]$
By denoting $\Psi : (\omega,x,t,\nu) \in \Omega \times Q \times \mathbb{R} \mapsto \eta'(\nu)g(\nu)\varphi(x,t) \in \mathbb{R}$, thanks to Proposition 1, $\Psi(.,u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q)$, and therefore $\Psi(.,u_{\mathcal{T},k})$ converges weakly (up to a subsequence denoted in the same way) in $L^2(\Omega \times Q)$ to an element called χ.
But, for any $\phi \in L^2(\Omega \times Q)$, $(\omega,x,t,\nu) \in \Omega \times Q \times \mathbb{R} \mapsto \phi(\omega,x,t)\Psi(\omega,x,t,\nu)$ is a Carathéodory function such that $(\phi\Psi(.,u_{\mathcal{T},k}))$ is uniformly integrable. It is based on the fact that for any subset H of $\Omega \times Q$,

$$\int_H |\phi\Psi(.,u_{\mathcal{T},k})|dxdtdP \leqslant C\big(\|\Psi(.,u_{\mathcal{T},k})\|_{L^2(H)}\big)\Big[\int_H |\phi|^2 dxdtdP\Big]^{1/2}.$$

Thus, at the limit,

$$\int_{\Omega\times Q} \chi\phi dxdtdP = \int_{\Omega\times Q} \int_0^1 \Psi(.,\mathbf{u}(.,\alpha))d\alpha\phi dxdtdP.$$

By identification, $\Psi(.,u_{\mathcal{T},k}) \to \int_0^1 \Psi(.,\mathbf{u}(.,\alpha))d\alpha$ weakly in $L^2(\Omega\times Q)$. Using now the linear continuity of the stochastic integral from $L^2(\Omega\times Q)$ to $L^2(\Omega\times\mathbb{R}^d)$, which implies the continuity for the weak topology:

$$\int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(t) \to \int_0^T \int_0^1 \eta'(\mathbf{u}(.,\alpha))g(\mathbf{u}(.,\alpha))\varphi d\alpha dW(t) \text{ weakly in } L^2(\Omega\times\mathbb{R}^d).$$

As $1_A 1_{B(0,R)} \in L^2(\Omega\times\mathbb{R}^d)$ one gets at the limit

$$E\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t)\Big] \to E\Big[1_A \int_0^T \int_{\mathbb{R}^d}\int_0^1 \eta'(\mathbf{u}(x,t,\alpha))g(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dxdW(t)\Big].$$

4. Study of $\frac{1}{2}E\Big[1_A \int_Q \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt\Big]$
Since $\Psi : (\omega,x,t,\nu) \in \Omega\times Q\times\mathbb{R} \mapsto \eta''(\nu)g^2(\nu)\varphi(x,t)1_A(\omega) \in \mathbb{R}$ is a Carathéodory function such that $\Psi(.,u_{\mathcal{T},k})$ is bounded in $L^2(\Omega\times Q)$, at the limit we get:

$$\frac{1}{2}E\Big[1_A \int_Q \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt\Big] \to \frac{1}{2}E\Big[1_A \int_Q\int_0^1 \eta''(\mathbf{u}(x,t,\alpha))g^2(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dxdt\Big].$$

Finally, by passing to the limit in Inequality (23), we obtain:
For any P-measurable set A, for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+(\mathbb{R}^d\times[0,T))$

$$\begin{aligned} 0 \leqslant\ & E\Big[1_A \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx\Big] + E\Big[1_A \int_Q\int_0^1 \eta(\mathbf{u}(x,t,\alpha))\varphi_t(x,t)d\alpha dxdt\Big] \\ & +E\Big[1_A \int_Q\int_0^1 F^\eta(\mathbf{u}(x,t,\alpha))\mathbf{v}.\nabla_x\varphi(x,t)d\alpha dxdt\Big] \\ & +E\Big[1_A \int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\mathbf{u}(x,t,\alpha))g(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dxdW(t)\Big] \\ & +E\Big[1_A \frac{1}{2}\int_Q\int_0^1 \eta''(\mathbf{u}(x,t,\alpha))g^2(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dxdt\Big]. \end{aligned}$$

Hence $\mathbf{u}$ is a measure-valued entropy solution in the sense of Definition 2. Thanks to Theorem 1, $\mathbf{u}$ is the unique stochastic entropy solution in the sense of Definition 1 and we denote it by u. Hence, all the sequence of approximate solution $u_{\mathcal{T},k}$ converges to u in $L^1_{loc}(\Omega\times Q)$. In addition, since $u_{\mathcal{T},k}$ is bounded in $L^2(\Omega\times Q)$, all the sequence converges in $L^p_{loc}(\Omega\times Q)$ for any $1 \leqslant p < 2$. ∎

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