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Symposium dedicated to the memory of

Professor Wacław Olszak

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PLASTICITY TODAY

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Edited by

A. SAWCZUK

International Centre for Mechanical Sciences, Udine, Italy

and

G. BIANCHI

Polytechnic of Milan, Italy

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Preface

Classical plasticity, as regards its experimental motivation, constitutive modelling, mathematical methods for dealing with its equations, and engineering applications of the theory, is a fairly well founded and established domain of mechanical sciences. The deformation theory of plasticity is properly understood as far as its drawbacks, applicability and usefulness are concerned. The flow theory is endowed with appropriate theorems and mathematical methods for dealing effectively with boundary value problems concerning various technological needs. Its particular field, the limit analysis theory, appears to be applicable and particularly useful in several domains of engineering. To be specific, we may add that classical plasticity is a mature science as to its:

(a) mathematical description of the rate independent material behaviour;
(b) methods of solutions regarding equations involving perfectly plastic material response;
(c) applications of the perfectly plastic model in metal forming and structural engineering, to mention two extremes.

However, the actual requirements of applied research need to be reflected upon and an attempt made to bring together the many facets which go to make up the present state of the domain roughly specified as plasticity. Problems of high pressure and high speed metal forming; of inelastic wave propagation; of the dynamics of vehicles and structures, naval, terrestrial and spatial; of structural behaviour under

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Local and Global Aspects in the Mathematical Theory of Plasticity†

PIERRE M. SUQUET

Université de Montpellier II and GRECO CNRS GDE, France

ABSTRACT

The attention is focused on three recent theories which contributed to the development of the mathematical theory of plasticity. The common point of these theories is the evidence or the use of the 'generalized standard' form of the encountered constitutive laws. Section 2 is devoted to homogenization. It demonstrates how elementary microscopic laws might give rise to standard macro-laws involving internal variables. The two next sections are devoted to a thermodynamical and mathematical discussion of these laws, as well as from a local standpoint or from a global one.

1. INTRODUCTION

The Mathematical Theory of Plasticity is the title of Hill's famous book written in the early 1950s [15]. Thirty years later, in 1983, after the remarkable and correlated development of computer sciences and of applied mathematics, after the emergence of continuum thermodynamics, this expression concerns more the field of convex analysis rather than that of Prandtl's nets. The past ten years have evidenced the benefit that mechanics could derive from collaborating with closely related fields and especially with applied analysis. The three sections of

† Part of this work was done while the author was at Mécanique Théorique, 4 place Jussieu, 75230 Paris Cédex 05, France

this paper refer to three theories elaborated in this spirit during the last ten years. Proceeding gradually from the particular case to the general one, they intend to demonstrate that theories with internal variables constitute a *rational approach* to continuum and structural mechanics.

Section 2 is devoted to *homogenization*, i.e. to theories which enable one to derive the global properties of highly heterogeneous media. The main point of the section goes as follows: consider two (or more) elastic perfectly plastic materials, aggregate them into a basic cell and repeat periodically this basic cell. Then the constitutive law of the mixture, derived by homogenization, requires the introduction of internal (micro-structural) variables.

Once the need for models with internal variables is emphasized we discuss, with the help of continuum thermodynamics, the theoretical *structure* of these models. The examples constitute the main point of Section 3 and show how this structure can be used to derive simple but efficient engineering models, endowed with the same mathematical characteristics.

In the last section the notion of standard law is generalized to structures where global variables (such as averaged strains, geometrical parameters of the structure, etc.) are under consideration. A rapid survey of the encountered variational problems obviates the mathematical common points in the discussion of the evolution of various systems or phenomena: plasticity, damage or rupture.

Since convexity is a common underlying feature of most of the paper, a small appendix is devoted to a non-polemical discussion of its 'hegemony'.

The three points under consideration here will be discussed in the context of infinitesimal strains. Another point, treated elsewhere in this Symposium, could have been an extension to finite strains. However, mathematical studies at finite strain are essentially as of now in the domain of 'Plasticity to-morrow'.

2. HOMOGENIZATION AND PLASTICITY

2.1. The Four Steps of Macro–Micromechanics

In the deterministic discussion of the overall properties of heterogeneous media the *first step* is to define a representative volume element (r.v.e) $\mathcal{V}$ small enough to distinguish the microscopic heterogeneities,

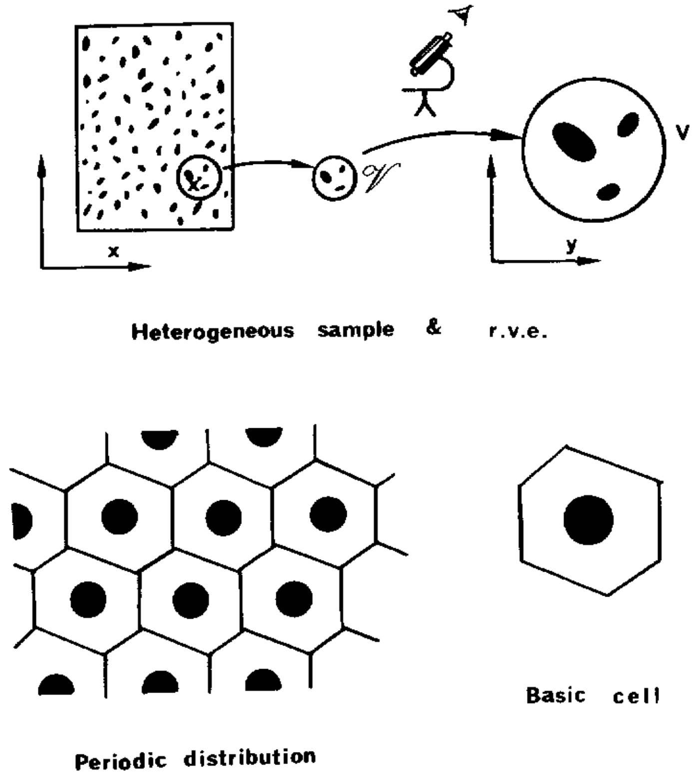

FIG. 1. Representative volume element: basic cell.

yet large enough to represent the overall behavior of the heterogeneous medium: a suitable rescaling (amplitude $1/\delta$) maps $\mathcal{V}$ onto an enlarged volume V where all the heterogeneities can easily be distinguished.

It turns out that the problem contains two scales: the 'macroscopic' scale x, on which all the macro-quantities depend and the 'microscopic'† scale y on which the micro-quantities depend. Consideration of the latter scale gives rise to highly oscillating fields in the heterogeneous body.

The *second step* of the analysis is the definition of macroscopic quantities from microscopic ones. This is achieved through an averag-

† Large enough to be in a position to apply continuum mechanics.

ing process: for a field f defined in the heterogeneous body we set

$$F(x)=\langle f\rangle(x)=\frac{1}{\operatorname{vol}(x+v)}\int_{x+v} f(\zeta)\,d\zeta=\frac{1}{\operatorname{vol}(v)}\int_v f(x+\xi)\,d\xi$$
$$=\frac{1}{\operatorname{vol} V}\int_V f(x+y\delta)\,dy=\frac{1}{\operatorname{vol}(V)}\int_V f(x,y)\,dy \tag{1}$$

where $f(x,y)=f(x+y\delta)$.

Most of the macro-quantities are the average in the preceding sense of the micro-ones.† It is the case for the stress and strain tensors, and for all the thermodynamic functions which are usually assumed to be additive functions: internal and free energy, entropy, dissipation, mass per unit volume . . .

$$\boldsymbol{\Sigma}=\langle\boldsymbol{\sigma}\rangle,\qquad \mathbf{E}=\langle\boldsymbol{\varepsilon}\rangle,\qquad \bar\rho=\langle\rho\rangle,\qquad \bar\rho W=\langle\rho w\rangle\ddagger\qquad D=\langle d\rangle,\ \bar\rho S=\langle\rho s\rangle \tag{2}$$

Remark 1: 1. If the constituents of the heterogeneous medium are not perfectly bonded (especially if voids or cracks are present) the averaging process is valid for *extended fields* defined on the defects [35].

2. We are not considering here purely statistical descriptions of the r.v.e. (Kröner [19], MacCoy [21]) or approximate ones (self-consistent models by Hill [16], Zaoui [37], Hashin's spheres model).

The *third step* in constructing a theory of homogenization is to define a *localization* procedure, i.e. a way of deriving microscopic fields from macroscopic ones. For this purpose the average relations (2) and the microscopic constitutive law must be taken into account together with other conditions arising from the equilibrium equations and from geometrical considerations. In order to derive the first condition we note that according to (1), the micro-stress field $\boldsymbol{\sigma}(x,y)$ satisfies

$$\operatorname{div}_y \boldsymbol{\sigma}(x,y)=\delta\operatorname{div}\boldsymbol{\sigma}(x+\delta y)=\delta(\rho\ddot{\mathbf{u}}-\rho\mathbf{f})\S \tag{3}$$

As far as the phenomena under consideration occur in a low frequency range (when compared to the size of the r.v.e.), the second member of (3) can be neglected. We obtain a microequilibrium equation

$$\operatorname{div}_y \boldsymbol{\sigma}=0 \tag{4}$$

† This assertion suffers counter examples: elastic coefficients, plastic strain, plastic work.

‡ Macro and micro fields are respectively denoted by capital or minus letters.

§ Div_y = divergence with respect to the y variable.

We are not yet in a position to determine the micro-fields $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ from the macro ones $\boldsymbol{\Sigma}$, $\mathbf{E}$, since we are missing suitable boundary conditions on ∂V. These boundary conditions arise from the geometrical arrangement of the heterogeneities. They should closely reproduce the boundary conditions of the r.v.e. V in the composite. Thus they should characterize the *in situ* state of stress and strain within the heterogeneous medium. They very often reduce to assuming that the micro-strain or stress is uniform on ∂V (hence equal to the macro-strain or stress); the boundary ∂V sometimes tends to infinity. Together with (4) this set of boundary conditions constitutes the *macro–micro localization conditions* (mml). It is highly desirable for mechanical reasons (average of the micro-work = macro-work) that the following *Hill's macro-homogeneity condition* holds true for all the fields $\boldsymbol{\sigma}^*$ and $\boldsymbol{\varepsilon}^*=\boldsymbol{\varepsilon}(\mathbf{u}^*)$ satisfying the mml conditions

$$\langle\boldsymbol{\sigma}^*\boldsymbol{\varepsilon}^*\rangle=\boldsymbol{\Sigma}^*\mathbf{E}^* \tag{5}$$

The *fourth* and last step consists of the homogenization procedure itself. The micro-constitutive law is known and the relationships between micro- and macro-fields have been established. It now remains to relate the macro-fields. This is easily done in a linear context but turns out to be a difficult task in non-linear problems.

We now describe in detail a special method of homogenization valid† for periodic media.

2.2. Periodic Media

The case of periodic media is of special interest in view of the large number of 'repetitive' structures encountered in industry. The choice of the r.v.e. is readily made: V is chosen to be the basic cell of the periodic structure (cf. Fig. 1). The localization conditions are directly derived from the geometry of the composite: away from the boundary of the sample the stress and strain fields conform at the micro-level to the periodic character of the geometry:

$$\boldsymbol{\sigma}(x,y)\boldsymbol{\varepsilon}(\mathbf{u}(x,y))\ \text{are } V\text{-periodic functions of } y\ddagger \tag{6}$$

From a mathematical standpoint u belongs to the space of fields with

† We emphasize that the method described can be rendered rigorous through an asymptotic analysis [2] [29] [35]. In this sense the theory is exact.

‡ We shall omit in the sequel the dependence of the micro-fields on the x variable.

Periodic Deformation. Elements of this space can be split into a linear and a periodic part

$$\mathrm{PD}(V)=\{\mathbf{u}\in H^1(V)^3,\ u_i=E_{ij}y_j+v_i,\ \mathbf{v}\in H^1_{\mathrm{per}}(V)^3\}\dagger \tag{7}$$

The constant tensor $\mathbf{E}$ occurring in (7) is precisely the macroscopic strain tensor associated with $\mathbf{u}$ by (2).

The stress field $\boldsymbol{\sigma}$ belongs to the following space of microscopically self-equilibrated fields:

$$S^{\circ}_{\mathrm{per}}(V)=\{\boldsymbol{\sigma}\in L^2(V)^9_s,\ \mathrm{div}_y\,\boldsymbol{\sigma}=0,\ \boldsymbol{\sigma}\,.\,\mathbf{n}(y)\text{ opposite on opposite sides of } V\}$$

Therefore the mml conditions for periodic media reduce to (4) and (6), i.e.

$$\boldsymbol{\sigma}\in S^{\circ}_{\mathrm{per}}(V),\qquad \mathbf{u}\in\mathrm{PD}(V) \tag{8}$$

Let us emphasize that these mml conditions satisfy Hill's macro-homogeneity condition.

2.2.1. *An Example: Elastic Perforated Media*

Let us denote by V^* the solid part of the basic cell V and by $\mathbf{a}(y)=(a_{ijkh}(y))$ its elastic coefficients. The main point of the procedure is the localization process: for a given macro-strain state $\mathbf{E}$ what is the induced micro-strain? The equations of the problem are:

$$\left.\begin{aligned}&\boldsymbol{\sigma}(y)=\mathbf{a}(y)\boldsymbol{\varepsilon}(u)\\&\boldsymbol{\sigma}\in S^{\circ}_{\mathrm{per}}(V),\qquad \mathbf{u}\in\mathrm{PD}(V),\ \text{macro-strain}=\mathbf{E}\end{aligned}\right\} \tag{9}$$

Using the definition of $\mathrm{PD}(V)$, we split $\mathbf{u}$ into a linear part $\mathbf{Ey}$ and a periodic part $\mathbf{v}$. Then by Hill's macrohomogeneity condition (5) we get

$$\langle\boldsymbol{\sigma\varepsilon}(\mathbf{v}^*)\rangle=\boldsymbol{\Sigma}\mathbf{E}^*=0 \text{ for every } \mathbf{v}^* \text{ in } H^1_{\mathrm{per}}(V)^3 \text{ since } \mathbf{E}^*=0$$

Therefore $\mathbf{v}$ is the solution of the following variational problem

$$\left.\begin{aligned}&\mathbf{v}\in H^1_{\mathrm{per}}(V)^3 \text{ and for every } \mathbf{v}^* \text{ in } H^1_{\mathrm{per}}(V)^3\\&\langle\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{v})\boldsymbol{\varepsilon}(\mathbf{v}^*)\rangle=-\langle\mathbf{aE}\boldsymbol{\varepsilon}(\mathbf{v}^*)\rangle\end{aligned}\right\} \tag{10}$$

The second member of (10) can be identified as a concentrated loading on the boundary of the heterogeneities. Because of the condition of periodicity, (10) is not a classical boundary value problem. However it is possible to prove that it admits a solution [6]. Since (10) is a linear problem with respect to E, its solution v can be decomposed along *the*

† $H^1_{\mathrm{per}}(V)=\{\text{periodic elements of } H^1(V)\}$.

basis formed with the 6 *following elementary solutions* $\boldsymbol{\chi}_{ij}$

$\mathbf{E}=E_{ij}\mathbf{I}_{ij}$ where $\mathbf{I}_{ij}$ is defined as

$$(I_{ij})_{kh}=\tfrac{1}{2}(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk})$$

$\mathbf{v}=E_{ij}\boldsymbol{\chi}_{ij}$ where $\boldsymbol{\chi}_{ij}$ is solution of (10) with $\mathbf{E}=\mathbf{I}_{ij}$

$$\boldsymbol{\varepsilon}(\mathbf{u})=\mathbf{E}+\boldsymbol{\varepsilon}(\mathbf{v})=(\mathbf{I}_{ij}+\boldsymbol{\varepsilon}(\boldsymbol{\chi}_{ij}))E_{ij}=(\mathbf{I}+\boldsymbol{\varepsilon}(\boldsymbol{\chi}))\mathbf{E} \tag{11}$$

This last equality completes *the localization process*: the micro-strain $\boldsymbol{\varepsilon}(\mathbf{u})$ can be computed in terms of the macro- one $\mathbf{E}$. The forthcoming example (see Fig. 2) gives a few numerically determined elementary fields $\boldsymbol{\chi}_{ij}$.

The fourth step (i.e. homogenization) is readily done through the averaging of the micro-constitutive law

$$\begin{aligned}\boldsymbol{\sigma}&=\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{u})=\mathbf{a}(\mathbf{I}+\boldsymbol{\varepsilon}(\boldsymbol{\chi}))\mathbf{E}\\\boldsymbol{\Sigma}&=\langle\boldsymbol{\sigma}\rangle=\langle\mathbf{a}(\mathbf{I}+\boldsymbol{\varepsilon}(\boldsymbol{\chi}))\rangle\mathbf{E}\\\mathbf{a}^{\mathrm{hom}}&=\langle\mathbf{a}(\mathbf{I}+\boldsymbol{\varepsilon}(\boldsymbol{\chi}))\rangle\end{aligned} \tag{12}$$

Remark 2: 1. Other mml conditions lead to the same type of elastic boundary value problem with any kind of 'classical' boundary conditions. Hill's equality (5) was the main point used there.

2. A localization with respect to the stress can be performed. For a given macro-stress $\boldsymbol{\Sigma}$ we can express the micro-stress $\boldsymbol{\sigma}(y)$ as a linear function of $\boldsymbol{\Sigma}$ [34]

$$\boldsymbol{\sigma}(y)=\mathbf{C}(y)\boldsymbol{\Sigma} \tag{13}$$

We are able to find the actual micro-stress state induced by a macro-one and to detect possible micro-stress concentrations. This is the fundamental goal of the localization process.

3. The example under consideration in Figs. 2, 3 and 4 originates in the work of Litewka and Sawczuk [20] where anisotropic damage was to be modelled. The basic cell, together with a few, numerically computed, strain and stress localizations are shown. The agreement with experimental data is good and shows how well the homogenization theory accounts for anisotropic behavior.

2.3. Rigid-plastic Constituents

In this section we assume that the constituents are rigid plastic and that they follow the normality rule

$$\boldsymbol{\sigma}(y)\in P(y),\qquad (\dot{\boldsymbol{\varepsilon}}(y),\ \boldsymbol{\sigma}^*-\boldsymbol{\sigma}(y))\leqslant 0\qquad \forall\,\boldsymbol{\sigma}^*\in P(y),\qquad \forall\, y\in V \tag{14}$$

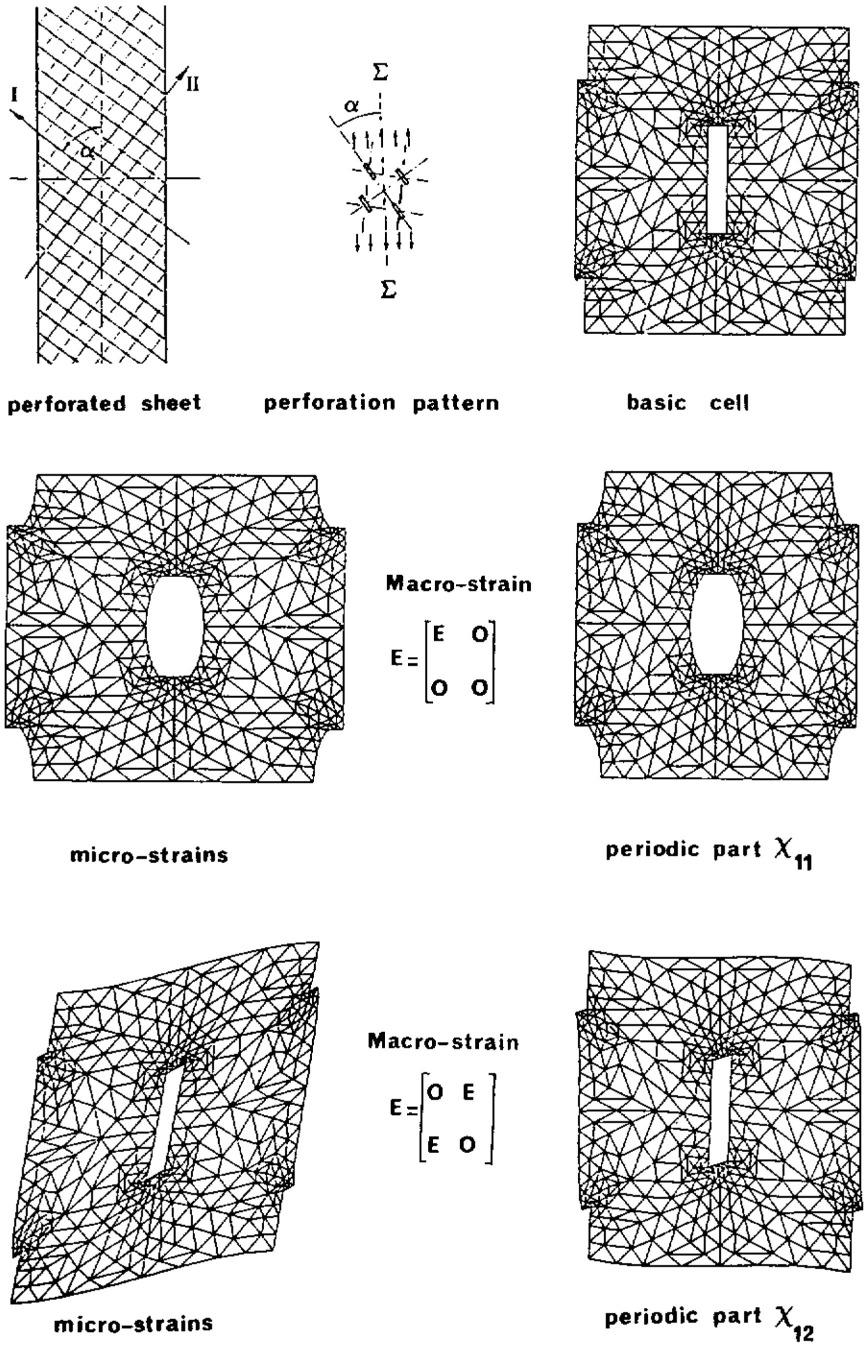

FIG. 2. Strain localization in perforated sheet.

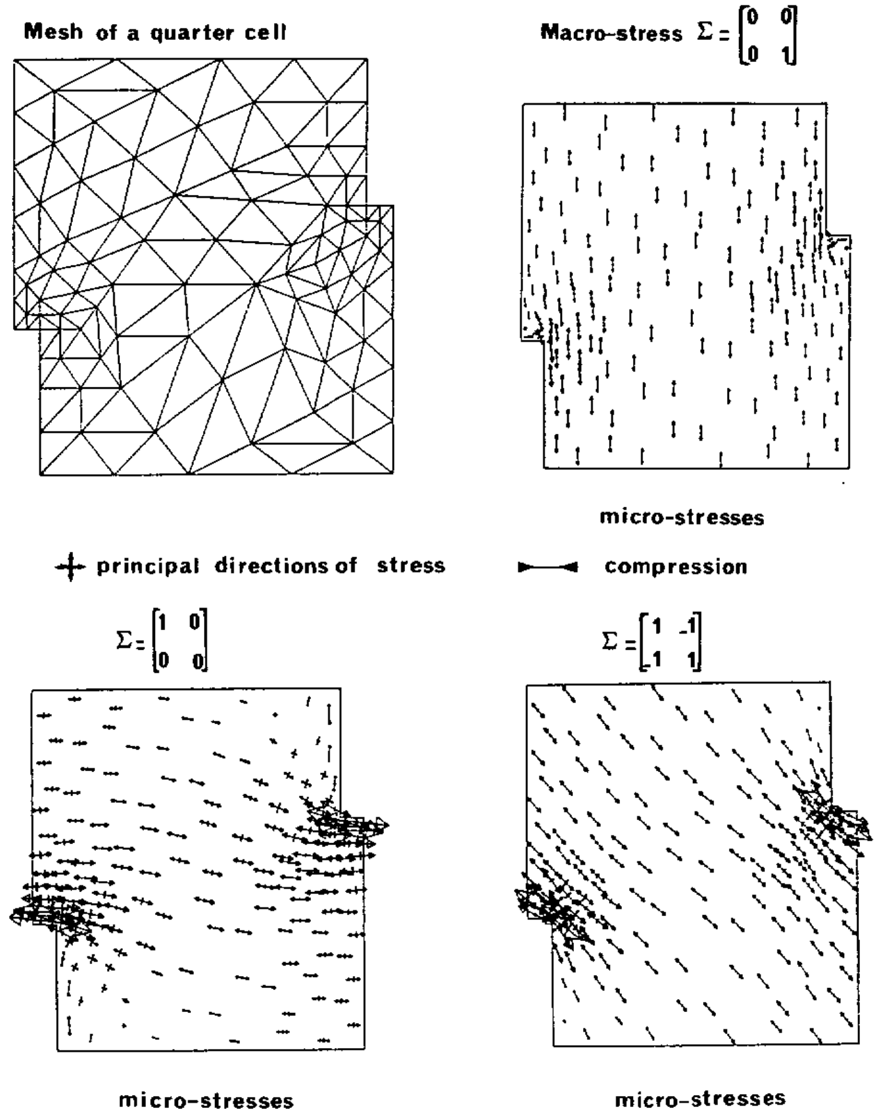

FIG. 3. Localization of stresses.

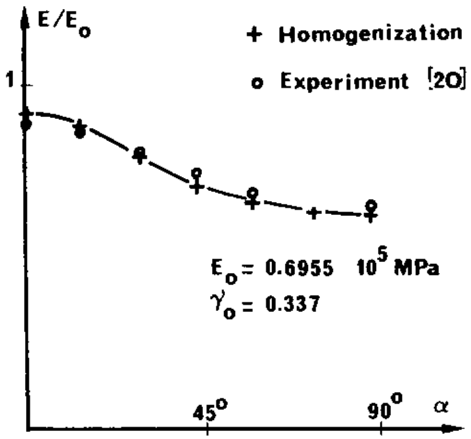

FIG. 4. Young's modulus $E(\alpha)$ in the direction of traction.

Since the micro-stress field is constrained, its average, the macro-stress $\boldsymbol{\Sigma}$, must be constrained too. Any admissible macro-stress $\boldsymbol{\Sigma}^*$ is necessarily related to a microscopic plastically admissible stress field satisfying the mml conditions. In the periodic case this leads to the following definition of the *macro-yield locus*

$$P^{\text{hom}} = \{\boldsymbol{\Sigma}^* \mid \exists \boldsymbol{\sigma}^*(y),\ \boldsymbol{\sigma}^* \in S^{\circ}_{\text{per}}(V),\ \langle\boldsymbol{\sigma}^*\rangle = \boldsymbol{\Sigma}^*,\ \boldsymbol{\sigma}^*(y) \in P(y)\ \forall y \in V\} \tag{15}$$

We claim that the normality rule holds true at the macro-level, i.e.

$$\boldsymbol{\Sigma} \in P^{\text{hom}}, \qquad (\dot{\mathbf{E}}, \boldsymbol{\Sigma}^* - \boldsymbol{\Sigma}) \leq 0 \qquad \forall\, \boldsymbol{\Sigma}^* \in P^{\text{hom}} \tag{16}$$

Indeed $\boldsymbol{\Sigma}$ clearly belongs to P^{hom}. Moreover to any $\boldsymbol{\Sigma}^*$ element of P^{hom} we can associate a micro-stress state $\boldsymbol{\sigma}^*(y)$ satisfying (15). Then by virtue of (14) and with the help of Hill's condition

$$0 \geq \langle \dot{\boldsymbol{\varepsilon}}(y),\ \boldsymbol{\sigma}^*(y) - \boldsymbol{\sigma}(y)\rangle = (\dot{\mathbf{E}}, \boldsymbol{\Sigma}^* - \boldsymbol{\Sigma}) \tag{17}$$

Remark 3: 1. Once more, Hill's condition plays a central role in the proof of the macro-normality while the periodicity is a technical point.

2. If we assume that the bonds between the constituents are plastic, i.e. that the stress vector at the interface S is constrained

$$\boldsymbol{\sigma} \cdot \mathbf{n}(z) \in P'(z) \qquad \forall z \in S \tag{18}$$

Then the macro-yield locus is given by

$$P^{\text{hom}} = \{\boldsymbol{\Sigma} \mid \exists \boldsymbol{\sigma}^*(y) \in S^{\circ}_{\text{per}}(V),\ \langle\boldsymbol{\sigma}^*\rangle = \boldsymbol{\Sigma}^*,\ \boldsymbol{\sigma}(y) \in P(y),\ \boldsymbol{\sigma} \cdot \mathbf{n}(z) \in P'(z) \forall y,\ z \in V \times S\}$$

3. If stratified materials are under consideration, proper use of the mml conditions permits the identification† of P^{hom} as:

$$P^{\text{hom}} = \{\boldsymbol{\Sigma} \mid \boldsymbol{\Sigma} = c_\alpha \boldsymbol{\Sigma}^\alpha,\ \boldsymbol{\Sigma}^\alpha_{i3} = \boldsymbol{\Sigma}_{i3},\ i = 1, 2, 3, \boldsymbol{\Sigma}^\alpha \in P^\alpha\}$$

where P^α and c_α respectively denote the yield locus and the concentration of each constituent (the directions y_1 and y_2 are the invariant directions of the stratification).

4. For the practical determination of P^{hom}, a limit analysis problem on the r.v.e. V is to be solved: the loading parameters are the 6 independent components of the macro-stress tensor. This limit load analysis can be performed either through a static approach (as suggested by (15)) or through a kinematic one: P^{hom} is equal to the intersection ($\dot{\mathbf{E}}$ varying) of the half spaces defined by

$$\left\{\boldsymbol{\Sigma} \text{ such that } \boldsymbol{\Sigma}\dot{\mathbf{E}} \leq \inf_{\substack{\mathbf{u} \in PD(V) \\ \text{macro-strain } \dot{\mathbf{E}}}} \langle \pi(y, \boldsymbol{\varepsilon}(\mathbf{u}))\rangle = \pi^{\text{hom}}(\dot{\mathbf{E}})\right\} \tag{19}$$

where $\pi(y, .)$ and π^{hom} are respectively the support functions of the sets $P(y)$ and P^{hom}. If $\boldsymbol{\Sigma}$ and $\dot{\mathbf{E}}$ denote the *actual* macro-stress and strain we have

$$\boldsymbol{\Sigma} = \frac{\partial \pi^{\text{hom}}}{\partial \dot{\mathbf{E}}}(\dot{\mathbf{E}}) \tag{20}$$

The preceding definition of P^{hom} using π^{hom} has been used in various settings. For instance, in the context of porous materials, Gurson [12] proposed to compute $\pi^{\text{hom}}(\dot{\mathbf{E}})$ by mean of a Riesz's approximation. V was chosen as a cylinder or a sphere and uniformity of the strain on ∂V was the assumed mml conditions. The velocity fields entering the Riesz approach were inspired from elastic solutions of the problem or from previous Rice and Tracey's work. Since the infimum in (19) is not necessarily obtained for these fields the value obtained for π^{hom} is overestimated and the set P^{hom} is approximated by the outside (upper

† This result was first established by De Buhan [5] in the two-dimensional case with another method of homogenization.

bound). In this sense Gurson's model overestimates the carrying capacity of porous metals.

5. We shall see in the following paragraphs a simple way of determining P^{hom} using an elasto–plastic problem.

2.4. Elasto–Plastic Constituents

Elasto–plastic constituents give rise to a more complex homogenization problem, since the residual micro-stresses due to the incompatibility of the anelastic micro-strains, have a micro-stored elastic energy, and therefore induce a hardening of the material. For a suitable description of this hardening an infinite number of internal variables (namely the whole set of the anelastic micro-strains) is required (see ref. 33). This kind of theoretical result is of little practical importance. However more specific information can be obtained on the macro-response to a specified loading and in the case of approximate models involving a finite number of internal variables.

Consider first the case of the macro-response to a specified loading. As a typical example we assume that the macro-strain rate $\dot{\mathbf{E}}$ can be kept constant (equal to $\dot{\mathbf{E}}^o$) and we investigate the macro-stress response. If we take the elastic part of the behavior to be linear, the problem of localization (determination of the micro-fields) becomes

$$\left.\begin{array}{l}\mathbf{A}(y)\dot{\boldsymbol{\sigma}}(y)+\dot{\boldsymbol{\varepsilon}}^P(y)=\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(y))=\boldsymbol{\varepsilon}(\dot{\mathbf{v}})+\dot{\mathbf{E}}^o\\ \boldsymbol{\sigma}\in S^o_{\text{per}}(V),\qquad \mathbf{v}\in H^1_{\text{per}}(V)^3\\ \boldsymbol{\sigma}\in\mathcal{P}=\{\boldsymbol{\sigma}\mid\boldsymbol{\sigma}(y)\in P(y)\forall y\}\end{array}\right\}\qquad(21)$$

Use of the maximal work principle yields the following variational formulation for $\boldsymbol{\sigma}$

$$\left.\begin{array}{l}\boldsymbol{\sigma}\in S^o_{\text{per}}(V)\cap\mathcal{P},\qquad \forall\boldsymbol{\sigma}^*\in S^o_{\text{per}}(V)\cap\mathcal{P}:\\ \langle\mathbf{A}\dot{\boldsymbol{\sigma}},\boldsymbol{\sigma}^*-\boldsymbol{\sigma}\rangle\geqslant\dot{\mathbf{E}}^o\langle\boldsymbol{\sigma}^*-\boldsymbol{\sigma}\rangle=(\dot{\mathbf{E}}^o,\boldsymbol{\Sigma}^*-\boldsymbol{\Sigma})\end{array}\right\}\qquad(22)$$

Solving (22) and averaging the micro-stress response gives the macro-response $\boldsymbol{\Sigma}(t)$.

If $\boldsymbol{\sigma}$ has a limit $\boldsymbol{\sigma}_\infty$,† as t tends to $+\infty$, we get

$$\left.\begin{array}{l}\boldsymbol{\Sigma}_\infty\in P^{\text{hom}},\qquad \forall\boldsymbol{\Sigma}^*\in P^{\text{hom}}:\\ 0\geqslant(\dot{\mathbf{E}}^o,\boldsymbol{\Sigma}^*-\boldsymbol{\Sigma}_\infty)\end{array}\right\}\qquad(23)$$

† Such a result can be proved under geometrical assumptions on $P(y)$ using Haraux's result [14] on asymptotic behavior of solutions of evolution equations.

Then $\boldsymbol{\Sigma}_\infty$ is on the boundary of P^{hom} and $\dot{\mathbf{E}}^o$ is an outer normal vector to P^{hom} at $\boldsymbol{\Sigma}_\infty$: we thus obtain another way of determining P^{hom} by solving an elasto–plastic problem with periodicity conditions and a fixed loading concentrated on the interfaces of the constituents (this last part can be easily deduced from the study of (21)).

2.5. Approximate Models

Knowledge of the actual macro-constitutive law requires knowledge of an infinite number of internal variables, namely the whole set of anelastic micro-strains. However some simplified models, with piecewise constant anelastic micro-strains can be proposed. For the sake of simplicity we shall assume in the sequel that the basic cell is made of two constituents, a matrix and a fiber, and that the anelastic micro-strains are constant on each of them:

$$\boldsymbol{\varepsilon}^P(y)=\mathbf{E}^P_m\theta_m(y)+\mathbf{E}^P_f\theta_f(y)\qquad(24)$$

where $\theta_m(y)=1$ in the matrix, 0 in the fiber (similar definition for θ_f). The micro-constitutive law is now

$$\boldsymbol{\varepsilon}(\mathbf{u})=\mathbf{E}+\boldsymbol{\varepsilon}(\mathbf{v})=\mathbf{A}\boldsymbol{\sigma}+\boldsymbol{\varepsilon}^P=\mathbf{A}\boldsymbol{\sigma}+\mathbf{E}^P_m\theta_m+\mathbf{E}^P_f\theta_f\qquad(25)$$

Setting $\mathbf{a}=\mathbf{A}^{-1}$ we get

$$\left.\begin{array}{l}\boldsymbol{\sigma}=\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{v})+\mathbf{a}\mathbf{E}-\mathbf{a}\mathbf{E}^P_m\theta_m-\mathbf{a}\mathbf{E}^P_f\theta_f\\ \boldsymbol{\sigma}\in S^o_{\text{per}}(V),\qquad \mathbf{v}\in H^1_{\text{per}}(V)^3\end{array}\right\}\qquad(26)$$

The problem (26) bears a strong resemblance to (10), and since it is linear with respect to $\mathbf{E}$, $\mathbf{E}^P_m$, $\mathbf{E}^P_f$ its solution $\mathbf{v}$ can be split into

$$\mathbf{v}=\mathbf{E}\boldsymbol{\chi}+\mathbf{E}^P_m\boldsymbol{\chi}^P_m+\mathbf{E}^P_f\boldsymbol{\chi}^P_f\qquad(27)$$

where $\boldsymbol{\chi}$ is the array of elastic localization fields defined previously by (10) and (11). $\boldsymbol{\chi}^P_m$ and $\boldsymbol{\chi}^P_f$ are solutions of

$$\left.\begin{array}{l}\boldsymbol{\chi}^P_m\in H^1_{\text{per}}(V)^3\text{ and for every }\mathbf{v}^*\in H^1_{\text{per}}(V)^3\\ \langle\mathbf{a}\boldsymbol{\varepsilon}(\boldsymbol{\chi}^P_m)\boldsymbol{\varepsilon}(\mathbf{v}^*)\rangle=\langle\mathbf{a}\theta_m\boldsymbol{\varepsilon}(\mathbf{v}^*)\rangle\end{array}\right\}\qquad(28)$$

(similar definition for $\boldsymbol{\chi}^P_f$).

We get the micro-strain as

$$\boldsymbol{\varepsilon}(\mathbf{u})=\mathbf{E}(\mathbf{I}+\boldsymbol{\varepsilon}(\boldsymbol{\chi}))+\mathbf{E}^P_m\boldsymbol{\varepsilon}(\boldsymbol{\chi}^P_m)+\mathbf{E}^P_f\boldsymbol{\varepsilon}(\boldsymbol{\chi}^P_f)\qquad(29)$$

We recall that the micro free energy amounts to

$$\rho w(\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}^P)=\tfrac{1}{2}\mathbf{a}(\boldsymbol{\varepsilon}(\mathbf{u})-\boldsymbol{\varepsilon}^P)(\boldsymbol{\varepsilon}(\mathbf{u})-\boldsymbol{\varepsilon}^P)$$

and that the macro free energy is the average of the micro one. Thus with the help of (24) and (29), the free energy can be expressed solely in terms of E, $E^{\mathrm{P}}_{\mathrm{m}}$, $E^{\mathrm{P}}_{\mathrm{f}}$:

$$\bar{\rho}W(\mathbf{E}, \mathbf{E}^{\mathrm{P}}_{\mathrm{m}}, \mathbf{E}^{\mathrm{P}}_{\mathrm{f}}) = \langle \rho w(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}^{\mathrm{P}}) \rangle \tag{30}$$

In the general case of heterogeneous elastic properties it should be noted that the macro-stress can be derived from (25) and (29). We obtain

$$\begin{aligned}\boldsymbol{\Sigma} &= \langle \boldsymbol{\sigma} \rangle = \langle \boldsymbol{\sigma}(\boldsymbol{\varepsilon}(\boldsymbol{\chi}) + \mathbf{I}) \rangle\dagger \\ &= \mathbf{a}^{\mathrm{hom}}\mathbf{E} - c_{\mathrm{f}}\langle \mathbf{a}(\boldsymbol{\varepsilon}(\boldsymbol{\chi}) + \mathbf{I}) \rangle_{\mathrm{f}} \mathbf{E}^{\mathrm{P}}_{\mathrm{f}} - c_{\mathrm{m}}\langle \mathbf{a}(\boldsymbol{\varepsilon}(\boldsymbol{\chi}) + \mathbf{I}) \rangle_{\mathrm{m}} \mathbf{E}^{\mathrm{P}}_{\mathrm{m}}\end{aligned}$$

where c_{f}, c_{m} are the concentrations of each constituent and where $\langle\,\rangle_{\mathrm{f}}$ denotes the average on the fiber. Therefore the anelastic macro-strain amounts to:

$$\begin{aligned}\mathbf{E}^{\mathrm{P}} &= \mathbf{A}^{\mathrm{hom}}[c_{\mathrm{f}}\langle \mathbf{a}(\boldsymbol{\varepsilon}(\boldsymbol{\chi}) + \mathbf{I}) \rangle_{\mathrm{f}} \mathbf{E}^{\mathrm{P}}_{\mathrm{f}} + c_{\mathrm{m}}\langle \mathbf{a}(\boldsymbol{\varepsilon}(\boldsymbol{\chi}) + \mathbf{I}) \rangle_{\mathrm{m}} \mathbf{E}^{\mathrm{P}}_{\mathrm{m}}] \\ &\neq c_{\mathrm{f}}\mathbf{E}^{\mathrm{P}}_{\mathrm{f}} + c_{\mathrm{m}}\mathbf{E}^{\mathrm{P}}_{\mathrm{m}}\end{aligned}$$

The anelastic macro-strain *is not the average of the micro one*. Indeed it can be proved that

$$\mathbf{E}^{\mathrm{P}} = \langle \mathbf{C}\boldsymbol{\varepsilon}^{\mathrm{P}} \rangle = \mathbf{A}^{\mathrm{hom}}\langle \mathbf{a}(\boldsymbol{\varepsilon}(\boldsymbol{\chi}) + \mathbf{I})\boldsymbol{\varepsilon}^{\mathrm{P}} \rangle \tag{31}$$

where $\mathbf{C}$ is the stress localization tensor defined in (13).

In the case of *homogeneous elastic coefficients* the array of elastic correctors $\boldsymbol{\chi}$ vanishes and the following simplifications occur:

$$\mathbf{E}^{\mathrm{P}} = c_{\mathrm{f}}\mathbf{E}^{\mathrm{P}}_{\mathrm{f}} + c_{\mathrm{m}}\mathbf{E}^{\mathrm{P}}_{\mathrm{m}} = \langle \boldsymbol{\varepsilon}^{\mathrm{P}} \rangle$$

$$\bar{\rho}W(\mathbf{E}, \mathbf{E}^{\mathrm{P}}_{\mathrm{m}}, \mathbf{E}^{\mathrm{P}}_{\mathrm{f}}) = \tfrac{1}{2}\mathbf{a}(\mathbf{E} - \mathbf{E}^{\mathrm{P}})(\mathbf{E} - \mathbf{E}^{\mathrm{P}}) + \left(\frac{c_{\mathrm{f}}c_{\mathrm{m}}}{2} - \alpha\right)\mathbf{a}(\mathbf{E}^{\mathrm{P}}_{\mathrm{m}} - \mathbf{E}^{\mathrm{P}}_{\mathrm{f}})(\mathbf{E}^{\mathrm{P}}_{\mathrm{m}} - \mathbf{E}^{\mathrm{P}}_{\mathrm{f}}) \tag{32}$$

where $\alpha = c_{\mathrm{f}}\langle \boldsymbol{\varepsilon}(\boldsymbol{\chi}^{\mathrm{P}}_{\mathrm{f}}) \rangle_{\mathrm{f}} = -c_{\mathrm{m}}\langle \boldsymbol{\varepsilon}(\boldsymbol{\chi}^{\mathrm{P}}_{\mathrm{f}}) \rangle_{\mathrm{m}}$

It should be noted that the *hardening* term of (32)

$$\left(\frac{c_{\mathrm{f}}c_{\mathrm{m}}}{2} - \alpha\right)\mathbf{a}(\mathbf{E}^{\mathrm{P}}_{\mathrm{m}} - \mathbf{E}^{\mathrm{P}}_{\mathrm{f}})(\mathbf{E}^{\mathrm{P}}_{\mathrm{m}} - \mathbf{E}^{\mathrm{P}}_{\mathrm{f}})$$

is always positive. This positive hardening is precisely the elastic energy of the stresses due to the difference of the anelastic strains in the fiber and in the matrix.

† We make use of Hill's condition.

Anticipating Section 3 we can observe that the macro state variables are $\mathbf{E}$, $\mathbf{E}^{\mathrm{P}}_{\mathrm{m}}$, $\mathbf{E}^{\mathrm{P}}_{\mathrm{f}}$. Computing the associated thermodynamical forces, namely the partial derivatives of $\bar{\rho}W$, yields:

$$\bar{\rho}\frac{\partial W}{\partial \mathbf{E}} = \boldsymbol{\Sigma}, \qquad \bar{\rho}\frac{\partial W}{\partial \mathbf{E}^{\mathrm{P}}_{\mathrm{m}}} = \langle \boldsymbol{\sigma} \rangle_{\mathrm{m}} \overset{\mathrm{def}}{=} \boldsymbol{\Sigma}_{\mathrm{m}}, \qquad \bar{\rho}\frac{\partial W}{\partial \mathbf{E}^{\mathrm{P}}_{\mathrm{f}}} = \langle \boldsymbol{\sigma} \rangle_{\mathrm{f}} = \boldsymbol{\Sigma}_{\mathrm{f}} \tag{33}$$

Moreover we derive the following relations from the normality law averaged on each constituent:

$$\left.\begin{aligned}(\dot{\mathbf{E}}^{\mathrm{P}}_{\mathrm{m}}, \boldsymbol{\Sigma}^* - \boldsymbol{\Sigma}_{\mathrm{m}}) &= \langle \dot{\boldsymbol{\varepsilon}}^{\mathrm{P}}, \boldsymbol{\Sigma}^* - \boldsymbol{\sigma} \rangle_{\mathrm{m}} \leq 0 \quad \text{for every } \boldsymbol{\Sigma}^* \in P_{\mathrm{m}} \\ (\dot{\mathbf{E}}^{\mathrm{P}}_{\mathrm{f}}, \boldsymbol{\Sigma}^{**} - \boldsymbol{\Sigma}_{\mathrm{f}}) &= \langle \dot{\boldsymbol{\varepsilon}}^{\mathrm{P}}, \boldsymbol{\Sigma}^{**} - \boldsymbol{\sigma} \rangle_{\mathrm{f}} \leq 0 \quad \text{for every } \boldsymbol{\Sigma}^{**} \in P_{\mathrm{f}}\end{aligned}\right\} \tag{34}$$

where P_{m}, P_{f} denote the yield loci of the constituents.

We are thus in the following position: starting from elementary (elastic perfectly plastic) constituents, we derived a macro-model including a larger number of internal variables. This model does not reduce to a classical one. However it satisfies the normality rule in a generalized sense. Indeed if we set

$$\boldsymbol{\alpha} = (\mathbf{E}^{\mathrm{P}}_{\mathrm{m}}, \mathbf{E}^{\mathrm{P}}_{\mathrm{f}}), \qquad \mathbf{A} = (\boldsymbol{\Sigma}_{\mathrm{m}}, \boldsymbol{\Sigma}_{\mathrm{f}}), \qquad P = P_{\mathrm{m}} \times P_{\mathrm{f}}$$

the macro-constitutive law becomes

$$\begin{aligned}&\boldsymbol{\Sigma} = \bar{\rho}\frac{\partial W}{\partial \mathbf{E}}(\mathbf{E}, \boldsymbol{\alpha}), \qquad \mathbf{A} = -\bar{\rho}\frac{\partial W}{\partial \boldsymbol{\alpha}}(\mathbf{E}, \boldsymbol{\alpha}) \quad \text{equations of state} \\ &\left.\begin{aligned}&\mathbf{A} \in P \text{ and for every } \mathbf{A}^* \text{ in } P: \\ &(\dot{\boldsymbol{\alpha}}, \mathbf{A}^* - \mathbf{A}) \leq 0\end{aligned}\right\} \quad \text{complementary laws}\end{aligned} \tag{35}$$

The macro-model is a *Generalized Standard Material* (GSM)

Remark 4: In this section the thermal coupling at the micro-scale has been neglected. In particular the question of deriving macro-thermodynamics (including a suitable definite of the macro temperature) from micro ones has been avoided. This important point is partly discussed in ref. 7 but remains largely open.

3. LOCAL GSM THEORY

The present section is devoted to the purely macroscopic study of Generalized Standard Materials (GSM) with the help of macroscopic thermodynamics.† The existence of such materials was established in the previous section where the occurrence in purely macro models of internal (micro-structural) variables was explained by macro–micro mechanics.

3.1. Generalized Standard Materials

The deformation of a continuous medium is a particular thermodynamical process which must be achieved in agreement with the two fundamental laws of thermodynamics. We assume here that the whole history of the medium is contained in the current value of a finite set of *state variables* $\boldsymbol{\chi}$ and that the thermostatic concepts of entropy s, temperature T, internal energy u, free energy w, extend to thermodynamical evolutions.‡ As far as the medium undergoes infinitesimal transformations,§ the state variables can be easily identified: they involve the infinitesimal strain ε and other physico-chemical variables $\boldsymbol{\alpha}$ (*internal variables*)

$$\boldsymbol{\chi}=(\boldsymbol{\varepsilon},\boldsymbol{\alpha}),\qquad u=u(\boldsymbol{\chi},s),\qquad w=w(\boldsymbol{\chi},T)\ldots \tag{36}$$

Applying the first and second law of thermodynamics yields the Clausius–Duhem inequality

$$d=\boldsymbol{\sigma}^{\mathrm{IR}}\dot{\boldsymbol{\varepsilon}}+\mathbf{A}\dot{\boldsymbol{\alpha}}-\mathbf{q}\frac{\boldsymbol{\nabla}T}{T}\geqslant 0 \tag{37}$$

where d denotes the total dissipation, $\mathbf{q}$ the heat flux, $\boldsymbol{\sigma}^{\mathrm{IR}}$ is the irreversible part of the stress while $\mathbf{A}$ is the array of the thermodynamical forces associated with $\boldsymbol{\alpha}$

$$\boldsymbol{\sigma}^{\mathrm{IR}}=\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{R}},\qquad \boldsymbol{\sigma}^{\mathrm{R}}=\rho\frac{\partial w}{\partial\boldsymbol{\varepsilon}},\qquad \mathbf{A}=-\rho\frac{\partial w}{\partial\boldsymbol{\alpha}}\quad\text{(equations of state)} \tag{38}$$

Assuming classically that the thermal and mechanical dissipations are decoupled yields their positivity in any real evolution

$$d_1=\boldsymbol{\sigma}^{\mathrm{IR}}\dot{\boldsymbol{\varepsilon}}+\mathbf{A}\dot{\boldsymbol{\alpha}}\geqslant 0\qquad d_2=-\mathbf{q}\frac{\nabla T}{T}\geqslant 0 \tag{39}$$

† Following the analysis of Halphen and Nguyen [13].
‡ This constitutes the basic assumption of the local accompanying state model [1] [9].
§ For finite strains see the accounts of refs. 13, 22 and 30.

The complementary constitutive laws relating $Y=(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}},\mathbf{q})$ and $\mathbf{X}=(\boldsymbol{\sigma}^{\mathrm{IR}},\mathbf{A},-\nabla T/T)$ must satisfy (39). In classical thermodynamics of irreversible processes [10] these laws are supposed to be linear and Onsager's relations imply the symmetry of the involved matrix:

$$Y_i=L_{ij}X_j\qquad L_{ij}=L_{ji} \tag{40}$$

A natural generalization of (40) consists in assuming that a mechanical and a thermal potential of dissipation $\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}})$, $\varphi_{\mathrm{th}}(\mathbf{q})$ exist, such that†

$$\boldsymbol{\sigma}^{\mathrm{IR}}=\frac{\partial\varphi}{\partial\dot{\boldsymbol{\varepsilon}}}(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}),\qquad \mathbf{A}=\frac{\partial\varphi}{\partial\dot{\boldsymbol{\alpha}}}(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}),\qquad -\frac{\nabla T}{T}=\frac{\partial\varphi_{\mathrm{th}}}{\partial\mathbf{q}}(\mathbf{q}) \tag{41}$$

If moreover the thermodynamic functions $u(\boldsymbol{\varepsilon},\boldsymbol{\alpha},s)$ and $\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}})$, $\varphi_{\mathrm{th}}(\mathbf{q})$ are positive convex functions of their arguments the material is said to be a Generalized Standard Material (GSM) [13].

Remark 5: 1. The positivity of d_1 and d_2 (4) directly follows from the convexity of φ and φ_{th}.
2. Taking $\varphi_{\mathrm{th}}=1/2k\ |\mathbf{q}|^2$ yields the Fourier's law:

$$\mathbf{q}=-\mathbf{k}\frac{\nabla T}{T}\simeq-\frac{\mathbf{k}}{T_0}\nabla\theta\quad\text{where}\quad \theta=T-T_0.$$

3. Introducing the Legendre–Fenchel transforms φ^* and φ^*_{th} of φ and φ_{th} one gets an equivalent formulation of (41)

$$\dot{\boldsymbol{\varepsilon}}=\frac{\partial\varphi^*}{\partial\boldsymbol{\sigma}^{\mathrm{IR}}}(\boldsymbol{\sigma}^{\mathrm{IR}},\mathbf{A}),\qquad \dot{\boldsymbol{\alpha}}=\frac{\partial\varphi^*}{\partial\mathbf{A}}(\boldsymbol{\sigma}^{\mathrm{IR}},\mathbf{A}),\qquad \mathbf{q}=\frac{\partial\varphi^*_{\mathrm{th}}}{\partial\left(-\frac{\nabla T}{T}\right)}\left(-\frac{\nabla T}{T}\right) \tag{42}$$

4. The rate independent materials constitute an important subclass of GSM. For these materials the law (41) does not depend on the scale of time, i.e. $\partial\varphi$ is homogeneous of degree 0 with respect to $(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}})$. Thus φ is homogeneous of degree 1 with respect to $(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}})$. A simple argument of convex analysis implies that φ^* is necessarily the indicator function of a convex set P. Furthermore the complementary laws (42)

† If the potentials are not differentiable, the following relations must be understood on the sense of subdifferentials (see any standard book on Convex Analysis).

expressed in a generalized way form the principle of maximal dissipation:†

There exists a closed convex set P such that

$$\left.\begin{array}{l}\varphi^*(\boldsymbol{\sigma}^{\mathrm{IR}},\mathbf{A})=0 \quad \text{if} \quad (\boldsymbol{\sigma}^{\mathrm{IR}},\mathbf{A})\in P,\ +\infty \quad \text{otherwise}\\ (\boldsymbol{\sigma}^{\mathrm{IR}},\mathbf{A})\in P \quad \text{and for every } (\boldsymbol{\sigma}^*,\mathbf{A}^*)\in P\\ (\dot{\boldsymbol{\varepsilon}},\boldsymbol{\sigma}^*-\boldsymbol{\sigma}^{\mathrm{IR}})+(\dot{\boldsymbol{\alpha}},\mathbf{A}^*-\mathbf{A})\leqslant 0\end{array}\right\} \tag{43}$$

P is the generalized plasticity locus.

Most of the time φ does not depend on $\dot{\boldsymbol{\varepsilon}}$ and (43) reduces to $\mathbf{A}\in P$ and for every $\mathbf{A}^*\in P$ $(\dot{\boldsymbol{\alpha}},\mathbf{A}^*-\mathbf{A})\leqslant 0$.

In the framework of GSM, rate independence is equivalent to the principle of maximal dissipation.

3.2. Examples

The GSM theory is confirmed, not only by its mathematical structure, but essentially by the number of classical or non-classical situations abiding by it. A few of them are briefly illustrated here through three (isothermal) examples.

3.2.1. *Damage of Ductile Metals* [27]

Ductile fracture in metals involves considerable damage at crack tips, through nucleation, growth and coalescence of voids initiated by inclusions. In order to account for this effect an internal variable describing the damage is introduced into a model of plasticity with hardening. For the sake of simplicity both effects (damage and hardening) are assumed to be isotropic. We set:

$$\begin{gathered}\boldsymbol{\alpha}=(\boldsymbol{\varepsilon}^{\mathrm{p}},D,p)\\ \rho w(\boldsymbol{\varepsilon},\boldsymbol{\alpha})=\tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{p}})\mathbf{a}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{p}})+h(p)+m(D)\end{gathered} \tag{44}$$

The equations of state are

$$\boldsymbol{\sigma}^{\mathrm{R}}=\mathbf{a}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{p}}),\qquad R=-\frac{\mathrm{d}h}{\mathrm{d}p}(p),\qquad Y=-\frac{\mathrm{d}m}{\mathrm{d}D}(D) \tag{45}$$

The effect of porosity on the elastic moduli has been neglected since we are mainly interested in the ductile behavior of the material. The presence of microvoids causing damage results in a sensitivity to pressure of the plasticity criterion. More precisely we assume that the plasticity locus is given in the space of generalized stresses as:

$$P=\{\boldsymbol{\sigma}^{\mathrm{IR}}=0,\ \mathbf{A}=(\boldsymbol{\sigma},Y,R)\ \text{satisfies: } J_2(\boldsymbol{\sigma})+\mathbf{R}+Yg(\sigma_{\mathrm{m}})\leqslant 0\}\dagger \tag{46}$$

The potential φ^* is the indicator function of P and the complementary laws are derived from the assumption of GSM

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{pD}}=\lambda\frac{\boldsymbol{\sigma}^{\mathrm{D}}}{J_2(\boldsymbol{\sigma})},\qquad \dot{\varepsilon}^{\mathrm{p}}_{\mathrm{m}}=\lambda Yg'(\sigma_{\mathrm{m}}),\qquad \dot{p}=\lambda,\qquad \dot{D}=\lambda g(\sigma_{\mathrm{m}}),\qquad \lambda\geqslant 0 \tag{47}$$

p is identified as the cumulated deviatoric plastic strain.

The form of g can be derived through elementary considerations. Because D is related to the volumic fraction of microvoids, it is thus related to the density ρ of the ductile material

$$D=D(\rho),\quad \text{i.e.}\quad \dot{D}=\frac{\mathrm{d}D}{\mathrm{d}\rho}(\rho)\dot{\rho}=\lambda g(\sigma_{\mathrm{m}}) \tag{48}$$

If we neglect the change of volume due to the elastic part of the strain, conservation of mass yields:

$$\dot{\rho}+3\rho\dot{\varepsilon}^{\mathrm{p}}_{\mathrm{m}}=0 \tag{49}$$

and, after due account of the complementary laws, we obtain:

$$\frac{g'(\sigma_{\mathrm{m}})}{g(\sigma_{\mathrm{m}})}=\frac{1}{Y}\frac{\dot{\varepsilon}^{\mathrm{p}}_{\mathrm{m}}}{\dot{D}}=-\frac{1}{3\rho\dfrac{\mathrm{d}D}{\mathrm{d}\rho}(\rho)Y} \tag{50}$$

But $Y=-(\mathrm{d}m/\mathrm{d}D)(D(\rho))$ is a function of ρ only. Thus the two sides of equality (50) are constant and homogeneous to the inverse of a stress (i.e. of the form c/σ_0) where σ_0 is the yield stress of the undamaged material. Integrating (50) yields:

$$g(\sigma_{\mathrm{m}})=\mu\exp\left(\frac{C\sigma_{\mathrm{m}}}{\sigma_0}\right)$$

Coming back to the complementary laws we get

$$\dot{D}=\lambda\mu\exp\left(\frac{C\sigma_{\mathrm{m}}}{\sigma_0}\right) \tag{51}$$

† Sometimes referred to as 'principle of maximal work'.

† $J_2(\boldsymbol{\sigma})=\sqrt{\sigma^{\mathrm{D}}_{ij}\sigma^{\mathrm{D}}_{ij}}$. $\boldsymbol{\sigma}^{\mathrm{D}}$ is the deviatoric part of $\boldsymbol{\sigma}$, σ_{m} is its spheric part.

The expansion rate of the damage (i.e. of the void fraction) is proportional to the exponential of the triaxiality σ_m/σ_0. It is remarkable that this result, established here through a completely macroscopic analysis, with the simple assumption (49), has been derived through a more precise micromechanical study by Rice and Tracey [26] for high triaxial stress states and by Gurson [12].† These latter studies also provide a precise estimate of C and ν.

The reader is referred to Rousselier [27] for an extension of the theory in the domain of finite strains, together with a study of the stability of the material described by this model.

3.2.2. *Viscoplasticity*

The previous example was concerned with rate independent laws. The effect of time can however be taken into account if one cares to consider viscoplastic laws. No account is given here for damage effects although it would be easy to supply one. Kinematic hardening is described by a tensorial internal variable $\boldsymbol{\beta}$. Therefore

$$\boldsymbol{\alpha} = (\boldsymbol{\varepsilon}^{\mathrm{p}}, \boldsymbol{\beta}, p)$$
$$\rho w(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \tfrac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}})\mathbf{a}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) + h(p) + k(\boldsymbol{\beta})$$

The state laws read as

$$\mathbf{A}_{\varepsilon^{\mathrm{p}}} = \boldsymbol{\sigma}^{\mathrm{R}} = \mathbf{a}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}), \qquad R = -\frac{\mathrm{d}h}{\mathrm{d}p}(p), \qquad \mathbf{B} = -D_{\boldsymbol{\beta}} k(\boldsymbol{\beta})$$

If we denote by P the preceding yield locus (46), the viscoplastic potential φ_n is

$$\varphi_n(\mathbf{A}) = \frac{\mu}{n+1}(j_{\mathrm{P}}(\mathbf{A}))^{n+1} = \frac{\mu}{n+1}\left(\frac{J_2(\boldsymbol{\sigma}+\mathbf{B})+R}{\sigma_0}\right)^{n+1}$$

where j_{P} is the gauge function of P.

The complementary laws are

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\mu}{\sigma_0}\left(\frac{J_2(\boldsymbol{\sigma}+\mathbf{B})+R}{\sigma_0}\right)^n \frac{\boldsymbol{\sigma}^{\mathrm{D}}+\mathbf{B}^{\mathrm{D}}}{J_2(\boldsymbol{\sigma}+\mathbf{B})}, \qquad \dot{\boldsymbol{\beta}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$$
$$\dot{p} = \frac{\mu}{\sigma_0}\left(\frac{J_2(\boldsymbol{\sigma}+\mathbf{B})+R}{\sigma_0}\right)^n \tag{52}$$

† Gurson established a dependence on cosh (σ_m/σ_0). But for high triaxial states (underlying assumption here) cosh $\simeq$ exp.

Fremond and Friaa [8] introduced a generalization of this law which turned out to be useful in limit analysis problems.

3.2.3. *Cyclic Viscoplasticity*

Under cyclic loadings the previous model fails to properly describe experimental data. Chaboche [4] obtained a good agreement with experimental data through the following modification of the law (52)

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\partial \varphi_n}{\partial \boldsymbol{\sigma}}(\mathbf{A}), \qquad \dot{p} = \frac{\partial \varphi_n}{\partial R}, \qquad \dot{\boldsymbol{\beta}} = \frac{\partial \varphi_n}{\partial \mathbf{B}} + \eta \mathbf{B}\dot{p} \tag{53}$$

The term $\eta \mathbf{B}\dot{p}$ is likely to alter the standard form of the law. However, if we admit a dependence of the potential on the state variables, it is still a standard law. Indeed† let us set

$$\tilde{\varphi}_n(\mathbf{A}, \boldsymbol{\alpha}) = \frac{\mu}{n+1}\left(\frac{J_2(\boldsymbol{\sigma}+\mathbf{B}) + R + \frac{\eta}{2}\mathbf{B}\,.\,\mathbf{B} - \frac{\eta}{2}D_{\boldsymbol{\beta}}k\,.\,D_{\boldsymbol{\beta}}k}{\sigma_0}\right)^{n+1}$$

The additional term $(\eta/2)\mathbf{B}\,.\,\mathbf{B} - (\eta/2)D_{\boldsymbol{\beta}}k\ D_{\boldsymbol{\beta}}k$ vanishes in all real evolutions, according to the equations of state. But the new complementary laws are

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\partial \tilde{\varphi}_n}{\partial \boldsymbol{\sigma}}(\mathbf{A}, \boldsymbol{\alpha}) = \frac{\partial \varphi_n}{\partial \boldsymbol{\sigma}}(\mathbf{A}), \qquad \dot{p} = \frac{\partial \tilde{\varphi}_n}{\partial R}(\mathbf{A}, \boldsymbol{\alpha}) = \frac{\partial \varphi_n}{\partial R}(\mathbf{A})$$
$$\dot{\boldsymbol{\beta}} = \frac{\partial \tilde{\varphi}_n}{\partial \mathbf{B}}(\mathbf{A}, \boldsymbol{\alpha}) = \frac{\partial \varphi_n}{\partial \mathbf{B}}(\mathbf{A}) + \frac{\mu\eta}{n+1}\left(\frac{J_2(\boldsymbol{\sigma}+\mathbf{B})+R}{\sigma_0}\right)^n.\,\mathbf{B} = \frac{\partial \varphi_n}{\partial \mathbf{B}}(\mathbf{A}) + \eta \mathbf{B}\dot{p}$$

which is exactly the desired law (53).

Comments. Chaboche's model leads us to address the following mathematical problem: a general form of complementary laws could be

$$\dot{\boldsymbol{\alpha}} = h(\mathbf{A}, \boldsymbol{\alpha})$$

Is it possible to add to h a suitable function $g(\mathbf{A}, \boldsymbol{\alpha})$, vanishing in every real evolution, i.e.

$$g\left(-\rho\frac{\partial W}{\partial \boldsymbol{\alpha}}, \boldsymbol{\alpha}\right) = 0$$

† This remark is due to J. L. Chaboche.

and giving to the law a standard form

$$h(\mathbf{A}, \boldsymbol{\alpha})+g(\mathbf{A}, \boldsymbol{\alpha})=\frac{\partial \varphi^*}{\partial \mathbf{A}}(\mathbf{A}, \boldsymbol{\alpha})?$$

4. GLOBAL GSM THEORY

4.1. Global Variables

When dealing with structures (i.e. with a whole body) it can be essential to consider state variables $\underset{\sim}{\alpha}$ which are not locally defined. For instance they can be geometrical parameters of the structure (think of optimal design or of cracked bodies), averages of local variables (think of shells or of homogenization), free boundaries, etc. Considering such *global variables* requires the introduction of a global thermodynamical formalism.

We assume that the system is endowed with a potential energy $F(\mathbf{u}, \underset{\sim}{\alpha})$ where $\mathbf{u}$ is the displacement field in the structure. We say that a set of global variables $\underset{\sim}{\alpha}$ is complete if the specification of this set of variables together with the specification of the loading suffices to determine the displacement field in the structure. For such a set of global variables we can express the total energy in terms of $\underset{\sim}{\alpha}$

$$\mathbf{u}=\mathbf{u}(\underset{\sim}{\alpha}), \qquad W(\underset{\sim}{\alpha})=F(\mathbf{u}(\underset{\sim}{\alpha}), \underset{\sim}{\alpha}) \tag{54}$$

We define global thermodynamic forces as

$$\underset{\sim}{A}=-\frac{\partial W}{\partial \underset{\sim}{\alpha}} \tag{55}$$

and the global GSM assumption is the following: there exists a convex functional $\Phi^*(\underset{\sim}{A})$ such that

$$\dot{\underset{\sim}{\alpha}}=\frac{\partial \Phi^*}{\partial \underset{\sim}{A}}(\underset{\sim}{A})\dagger \tag{56}$$

Once more we shall emphasize the role played by materials that satisfy the principle of maximal dissipation, for which Φ^* is the indicator function of a convex set P:

$$\underset{\sim}{A} \in P \quad \text{and for every} \quad \underset{\sim}{A}^* \in P, \quad (\dot{\underset{\sim}{\alpha}}, \underset{\sim}{A}^*-\underset{\sim}{A}) \leqslant 0 \tag{57}$$

† This equality is to be understood in the sense of subdifferentials for non-differentiable Φ^*.

4.1.1. *Examples*

(a) *Cracks* (Fig. 5(a)). Consider in a two-dimensional context a linear crack of length $\underset{\sim}{\alpha}$ in an elastic body. $\underset{\sim}{\alpha}$ is a global state variable of the structure since, once $\underset{\sim}{\alpha}$ is known and the loading is specified the displacement field is derived by solving a classical elastic problem

$$W(\underset{\sim}{\alpha})=\underset{\substack{\mathbf{u}^*=0 \text{ on } \Gamma_0 \\ [u_2^*]\geqslant 0 \text{ on } \Gamma_\alpha}}{\operatorname{Min}} \int_{\Omega_\alpha} \tfrac{1}{2}\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{u}^*)\boldsymbol{\varepsilon}(\mathbf{u}^*)\, dx-\int_{\Gamma_1} \mathbf{F}\mathbf{u}^*\, ds \tag{58}$$

The force $\underset{\sim}{A}$ is the energy release rate. If we define a set P as

$$P=\{\underset{\sim}{A}^* \mid \underset{\sim}{A}^* \leqslant \gamma\} \tag{59}$$

the corresponding rate independent law is a law of brittle fracture.

(b) *Damage* (Fig. 5(b) and (c)). Bui and Erlacher [3] introduced a

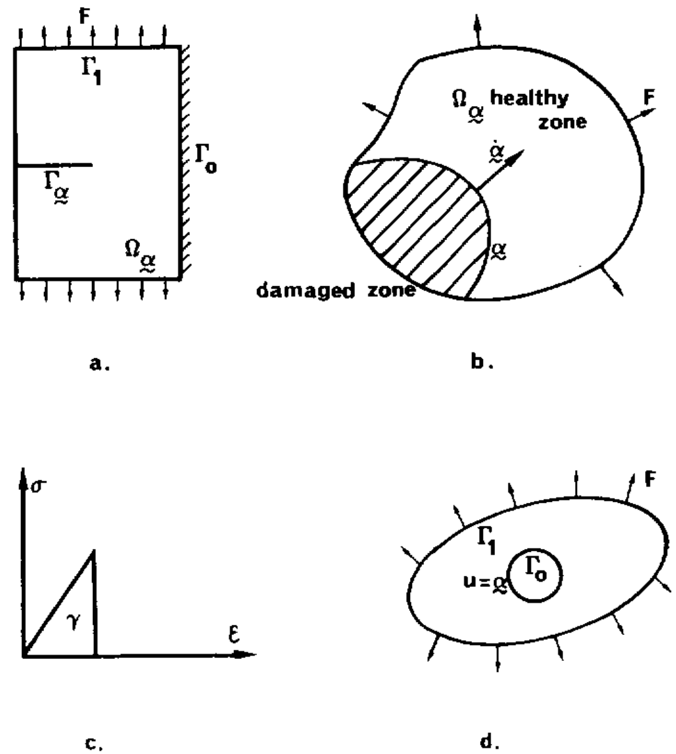

FIG. 5. Global variables.

law of total damage which goes as follows: Consider a structure which contains a 'healthy' linearly elastic zone and a totally damaged zone which is unable to support any load ($\boldsymbol{\sigma}=0$ in it). The damage front $\underset{\sim}{\boldsymbol{\alpha}}$ (free boundary) is a global variable of the structure since, once it is specified the displacement field in the healthy zone is derived as the solution of a classical elastic problem, while the front $\underset{\sim}{\boldsymbol{\alpha}}$ is stress-free

$$W(\underset{\sim}{\boldsymbol{\alpha}}) = \underset{\mathbf{u}^*}{\mathrm{Min}} \int_{\Omega_\alpha} \tfrac{1}{2}\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{u}^*)\boldsymbol{\varepsilon}(\mathbf{u}^*)\,\mathrm{d}x - \int_{\Gamma_1} \mathbf{F}\mathbf{u}^*\,\mathrm{d}s$$

The force $\underset{\sim}{\mathbf{A}}$ is the density of elastic energy on the damage front, $\dot{\underset{\sim}{\boldsymbol{\alpha}}}$ is the normal velocity of displacement of the front. If we choose a convex set P in the form (59) the rate independent law (57) yields

$$\begin{aligned} &\dot{\underset{\sim}{\boldsymbol{\alpha}}}(x) = 0 \quad \text{if} \quad \tfrac{1}{2}\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{u}(x))\boldsymbol{\varepsilon}(\mathbf{u}(x)) < \gamma \\ &\dot{\underset{\sim}{\boldsymbol{\alpha}}}(x) = \lambda\mathbf{n}(x)\ \lambda \geq 0 \quad \text{if} \quad \tfrac{1}{2}\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{u}(x))\boldsymbol{\varepsilon}(\mathbf{u}(x)) = \gamma \end{aligned} \tag{60}$$

In the one-dimensional case, the stress–strain relation obtained is plotted in Fig. 5(c).

(c) *Homogenization.* Considering the basic cell as a (micro)structure we claim that in the approximate model under consideration in Section 2.5, the set $\underset{\sim}{\boldsymbol{\alpha}} = (\mathbf{E}, \mathbf{E}^{\mathrm{P}}_{\mathrm{m}}, \mathbf{E}^{\mathrm{P}}_{\mathrm{f}})$ is a global state variable for the (micro)structure. Indeed, once it has been specified we are able to derive the displacement in the (micro)structure by (29), and the global energy by (30). The set of forces $\underset{\sim}{\mathbf{A}}$ is $(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\mathrm{m}}, \boldsymbol{\Sigma}_{\mathrm{f}})$ and the rate independent law (57) is (34). We can consider additional geometrical parameters of the microstructure: porosity of voids, surface of cracks, shape of inclusions. These micro-global variables become, through the homogenization process, macro-local variables.

(d) *Friction.* With the notation of Fig. 5(d), the displacement field $\underset{\sim}{\boldsymbol{\alpha}}$ on the part Γ_0 of $\partial\Omega$ is a global variable for an elastic body: once $\underset{\sim}{\boldsymbol{\alpha}}$ is specified, the displacement in the structure is the solution of

$$W(\underset{\sim}{\boldsymbol{\alpha}}) = \underset{\mathbf{u}^* = \underset{\sim}{\boldsymbol{\alpha}} \text{ on } \Gamma_0}{\mathrm{Min}} \int_{\Omega} \tfrac{1}{2}\mathbf{a}\boldsymbol{\varepsilon}(\mathbf{u}^*)\boldsymbol{\varepsilon}(\mathbf{u}^*)\,\mathrm{d}x - \int_{\Gamma_1} \mathbf{F}\mathbf{u}^*\,\mathrm{d}s \tag{61}$$

The force associated with $\underset{\sim}{\boldsymbol{\alpha}}$ is $-\boldsymbol{\sigma}\mathbf{n}\,|_{\Gamma_0}$. Let us set

$$P = \{\underset{\sim}{\mathbf{A}}^* \mid |\underset{\sim}{\mathbf{A}}^*_{\mathrm{T}}| \leq k\}\dagger$$

† $\underset{\sim}{\mathbf{A}}_{\mathrm{T}}$ is the tangential part of the vector $\underset{\sim}{\mathbf{A}}$.

The rate independent law (57) yields Tresca's friction law $\dot{u}_{\mathrm{N}} = 0$, $\dot{\mathbf{u}}_{\mathrm{T}} = 0$ if $|(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}| < k$, $\dot{\mathbf{u}}_{\mathrm{T}} = -\lambda(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}$, $\lambda \geq 0$ if $|(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}| = k$.

4.2. Evolution Problems for GSM

The set of eqns. (55) and (56) can be written:

$$\dot{\underset{\sim}{\boldsymbol{\alpha}}} - \frac{\partial \Phi^*}{\partial \underset{\sim}{\mathbf{A}}}\left(-\frac{\partial W}{\partial \underset{\sim}{\boldsymbol{\alpha}}}(\underset{\sim}{\boldsymbol{\alpha}}, t)\right) = 0, \qquad \underset{\sim}{\boldsymbol{\alpha}}(0) = \boldsymbol{\alpha}_0 \tag{62}$$

A similar evolution equation can be stated for $\underset{\sim}{\mathbf{A}}$. It is to be noticed that the equations governing the evolution of phenomena as different as plasticity, damage, brittle fracture, friction, etc., are similar. The discussion of (62) in a general context has not been done and is an open problem. Let us just mention two simple cases where existence and uniqueness of a solution of (62) can be easily established:

(a) If $\dfrac{\partial \Phi^*}{\partial \underset{\sim}{\mathbf{A}}}$ and $\dfrac{\partial W}{\partial \underset{\sim}{\boldsymbol{\alpha}}}$ are Lipschitz operators (this is the case of viscoelasticity) eqn. (62) reduces to a differential equation. Global existence and uniqueness of a solution are proved with the help of the Cauchy–Lipschitz theorem.

(b) If $\underset{\sim}{\boldsymbol{\alpha}} \to \dfrac{\partial \Phi^*}{\partial \underset{\sim}{\mathbf{A}}}\left(-\dfrac{\partial W}{\partial \underset{\sim}{\boldsymbol{\alpha}}}(\underset{\sim}{\boldsymbol{\alpha}}, t)\right)$ is a continuous coercive operator from a reflexive Banach space V into its dual space V' where $V \subsetneq H \subsetneq V'$ (H is a Hilbert space), and provided that it exhibits a smooth dependence on t, a standard theorem on evolution equations ensures the global existence and the uniqueness of a solution of (62).

Consider for instance a linearly elastic solid lying on a support Γ_0 with viscous friction (notation of Fig. 4):

$$\dot{u}_{\mathrm{N}} = 0, \qquad \dot{\mathbf{u}}_{\mathrm{T}} = -\lambda(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}} \quad \text{on} \quad \Gamma_0 \tag{63}$$

Considering $\underset{\sim}{\boldsymbol{\alpha}} = \mathbf{u}_{\mathrm{T}}$ as a global variable, we identify the associated force as $\underset{\sim}{\mathbf{A}} = (-\boldsymbol{\sigma}(\underset{\sim}{\boldsymbol{\alpha}})\,.\,\mathbf{n})_{\mathrm{T}}\,|_{\Gamma_0}$ where $\boldsymbol{\sigma}(\underset{\sim}{\boldsymbol{\alpha}})$ is derived from $\mathbf{u}(\underset{\sim}{\boldsymbol{\alpha}})$ by the elastic constitutive law. $\mathbf{u}(\underset{\sim}{\boldsymbol{\alpha}})$ is obtained as the solution of an elastic problem similar to (61).†

Therefore the viscous friction law is a global GSM law with

$$\Phi^*(\underset{\sim}{\mathbf{A}}) = \frac{\lambda}{2}|\underset{\sim}{\mathbf{A}}|^2$$

† The constraint for the minimization is now $u_{\mathrm{N}} = 0$, $\mathbf{u}_{\mathrm{T}} = \underset{\sim}{\boldsymbol{\alpha}}$.

Setting $V=H^{1/2}(\Gamma_0)^2$, $H=L^2(\Gamma_0)^2$, $V'=H^{-1/2}(\Gamma_0)^2$, we see that

$$\frac{\partial\Phi^*}{\partial\underset{\sim}{\mathbf{A}}}\left(-\frac{\partial W}{\partial\underset{\sim}{\alpha}}(\underset{\sim}{\alpha},t)\right)=-\lambda(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}(\underset{\sim}{\alpha})$$

It can be easily shown that

$$|(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}(\underset{\sim}{\alpha}_1)-(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}(\underset{\sim}{\alpha}_2)|_{V'}\leq C\,|\underset{\sim}{\alpha}_1-\underset{\sim}{\alpha}_2|_V$$
$$((\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}(\underset{\sim}{\alpha}_1)-(\boldsymbol{\sigma}\,.\,\mathbf{n})_{\mathrm{T}}(\underset{\sim}{\alpha}_2),\underset{\sim}{\alpha}_1-\underset{\sim}{\alpha}_2)_{V',V}\geq C'\,|\underset{\sim}{\alpha}_1-\underset{\sim}{\alpha}_2|_V^2$$

which establishes the continuity and coercivity of the operator under consideration. If the external loading (F) depends smoothly on t, the above mentioned theorem ensures existence and uniqueness of a solution for viscous friction.

Specific attention must be paid to materials that satisfy the principle of maximal dissipation (57):

$$\dot{\underset{\sim}{\alpha}}\in\partial I_{\mathrm{P}}(\underset{\sim}{\mathbf{A}})\tag{64}$$

The evolution problem, which now consists of the set of equations (55) and (64) is a highly non-linear one, and as such it remains unsolved† in the general case.

Debating of the existence of the rates‡ $\dot{\underset{\sim}{\alpha}}$ and $\dot{\underset{\sim}{\mathbf{A}}}$ is an easier task.

4.3. Variational Principle for $\dot{\underset{\sim}{\alpha}}$ [25]

$\dot{\underset{\sim}{\alpha}}$ is the solution of the following variational problem

$$\left.\begin{array}{l}\dot{\underset{\sim}{\alpha}}\in\partial I_{\mathrm{P}}(\underset{\sim}{\mathbf{A}})\quad\text{and for every}\quad\underset{\sim}{\alpha}^*\in\partial I_{\mathrm{P}}(\underset{\sim}{\mathbf{A}})\\ \left(\dfrac{\partial^2W}{\partial\underset{\sim}{\alpha}^2}(\underset{\sim}{\alpha},t)\dot{\underset{\sim}{\alpha}},\underset{\sim}{\alpha}^*-\dot{\underset{\sim}{\alpha}}\right)\geq-\left(\dfrac{\partial^2W}{\partial\underset{\sim}{\alpha}\,\partial t}(\underset{\sim}{\alpha},t),\underset{\sim}{\alpha}^*-\dot{\underset{\sim}{\alpha}}\right)\end{array}\right\}\tag{65}$$

and therefore has the following variational property:

$\dot{\underset{\sim}{\alpha}}$ minimizes among all admissible rates $\underset{\sim}{\alpha}^*\in\partial I_{\mathrm{P}}(\underset{\sim}{\mathbf{A}})$ the functional

$$\frac{1}{2}\frac{\partial^2W}{\partial\underset{\sim}{\alpha}^2}(\underset{\sim}{\alpha},t)\underset{\sim}{\alpha}^*\underset{\sim}{\alpha}^*+\frac{\partial^2W}{\partial\underset{\sim}{\alpha}\,\partial t}(\underset{\sim}{\alpha},t)\underset{\sim}{\alpha}^*\tag{66}$$

Proof [24] [25]. By the principle of maximal dissipation

$$(\underset{\sim}{\mathbf{A}}(t)-\underset{\sim}{\mathbf{A}}^*,\underset{\sim}{\alpha}^*)\geq0\quad\text{for every}\quad\underset{\sim}{\alpha}^*\in\partial I_{\mathrm{p}}(\underset{\sim}{\mathbf{A}}),\qquad\underset{\sim}{\mathbf{A}}^*\in P\tag{67}$$

† Moreau [23] and Brezis proved the existence and the uniqueness of a solution for $\underset{\sim}{\mathbf{A}}$, provided that $W(\underset{\sim}{\alpha})$ is a quadratic function.

‡ Since $\underset{\sim}{\alpha}(t)$ is not necessarily derivable with respect to t, the rates $\dot{\underset{\sim}{\alpha}}$, $\dot{\underset{\sim}{\mathbf{A}}}$ are the *right* derivatives $\dfrac{\mathrm{d}^+\underset{\sim}{\alpha}}{\mathrm{d}t},\dfrac{\mathrm{d}^+\underset{\sim}{\mathbf{A}}}{\mathrm{d}t}$ ('future' rates).

Taking $\underset{\sim}{\mathbf{A}}^*=\underset{\sim}{\mathbf{A}}(t+\mathrm{d}t)$ in (67) and dividing by $\mathrm{d}t$ yields

$$(\dot{\underset{\sim}{\mathbf{A}}},\underset{\sim}{\alpha}^*)\leq0\quad\text{for every}\quad\underset{\sim}{\alpha}^*\in\partial I_{\mathrm{P}}(\underset{\sim}{\mathbf{A}})$$

On the other hand it can be shown [24] that

$$(\dot{\underset{\sim}{\mathbf{A}}},\dot{\underset{\sim}{\alpha}})=0$$

Therefore

$$0\geq(\dot{\underset{\sim}{\mathbf{A}}},\underset{\sim}{\alpha}^*-\dot{\underset{\sim}{\alpha}})=-\left(\frac{\partial^2W}{\partial\underset{\sim}{\alpha}^2}(\underset{\sim}{\alpha},t)\dot{\underset{\sim}{\alpha}},\underset{\sim}{\alpha}^*-\dot{\underset{\sim}{\alpha}}\right)-\left(\frac{\partial^2W}{\partial\underset{\sim}{\alpha}\,\partial t}(\underset{\sim}{\alpha},t),\underset{\sim}{\alpha}^*-\dot{\underset{\sim}{\alpha}}\right)$$

which completes the proof of (66).

4.4. Variational Principle for $\dot{\mathbf{A}}$

In most examples $\underset{\sim}{\mathbf{A}}$ satisfies a linear constraint of the type:

$$\underset{\sim}{\mathbf{A}}\in S(f)=\{\underset{\sim}{\mathbf{A}}^*\mid\mathrm{L}(\underset{\sim}{\mathbf{A}}^*)=\mathbf{f}\}\tag{68}$$

where $\mathbf{f}$ is a constant quantity, L is a linear operator. Therefore the use of a Lagrange multiplier accounting for (68) permits inversion of the relation (55) between $\underset{\sim}{\alpha}$ and $\underset{\sim}{\mathbf{A}}$:

$$\underset{\sim}{\alpha}=\frac{\partial W^*}{\partial\underset{\sim}{\mathbf{A}}}(-\underset{\sim}{\mathbf{A}},t)+\underset{\sim}{\mathbf{e}}(t)$$

where W^* is the Legendre transform of W, e is the Lagrange multiplier orthogonal to $S(\mathbf{0})$ (take $f=0$ in (68)). Thus

$$\dot{\underset{\sim}{\alpha}}=-\frac{\partial^2W^*}{\partial\underset{\sim}{\mathbf{A}}^2}\dot{\underset{\sim}{\mathbf{A}}}+\frac{\partial^2W^*}{\partial\underset{\sim}{\mathbf{A}}\,\partial t}+\dot{\underset{\sim}{\mathbf{e}}}$$

It is to be noticed that because of the linearity of L, $\dot{\underset{\sim}{\mathbf{A}}}$ belongs to $S(\dot{\mathbf{f}})$ and that $\underset{\sim}{\mathbf{e}}$ is orthogonal to $S(\mathbf{0})$. Moreover, since $\underset{\sim}{\mathbf{A}}$ must belong to P, $\dot{\underset{\sim}{\mathbf{A}}}$ must belong to the projecting cone $\dot{P}(\underset{\sim}{\mathbf{A}})$. Thus we get the following variational formulation for $\dot{\underset{\sim}{\mathbf{A}}}$

$$\begin{array}{c}\dot{\underset{\sim}{\mathbf{A}}}\in\dot{P}(\underset{\sim}{\mathbf{A}})\cap S(\dot{\mathbf{f}})\quad\text{and for every}\quad\underset{\sim}{\mathbf{A}}^*\in\dot{P}(\mathbf{A})\cap S(\dot{\mathbf{f}}):\\ \left(\dfrac{\partial^2W^*}{\partial\underset{\sim}{\mathbf{A}}^2}(-\underset{\sim}{\mathbf{A}},t)\dot{\underset{\sim}{\mathbf{A}}},\underset{\sim}{\mathbf{A}}^*-\dot{\underset{\sim}{\mathbf{A}}}\right)\geq\left(\dfrac{\partial^2W^*}{\partial\underset{\sim}{\mathbf{A}}\,\partial t}(-\underset{\sim}{\mathbf{A}},t),\underset{\sim}{\mathbf{A}}^*-\dot{\underset{\sim}{\mathbf{A}}}\right)\end{array}\tag{69}$$

Therefore $\dot{\underset{\sim}{\mathbf{A}}}$ has the following variational property: $\dot{\underset{\sim}{\mathbf{A}}}$ minimizes among all admissible rates $\underset{\sim}{\mathbf{A}}^*\in\dot{P}(\underset{\sim}{\mathbf{A}})\cap S(\dot{\mathbf{f}})$ the following functional

$$\frac{1}{2}\frac{\partial^2W^*}{\partial\underset{\sim}{\mathbf{A}}^2}(-\underset{\sim}{\mathbf{A}},t)\underset{\sim}{\mathbf{A}}^*\underset{\sim}{\mathbf{A}}^*-\frac{\partial^2W^*}{\partial\underset{\sim}{\mathbf{A}}\,\partial t}(-\underset{\sim}{\mathbf{A}},t)\underset{\sim}{\mathbf{A}}^*\tag{70}$$

4.4.1. *Example*

Consider an elasto–plastic body clamped on its boundary and submitted to a volume loading f. The field of anelastic strains $\underset{\sim}{\alpha} = (\varepsilon^p(x))_{x\in\Omega}$ is a global variable for the structure since, once it is known, the displacement in the whole body is known as the solution of

$$W(\underset{\sim}{\alpha}) = \underset{\mathbf{u}^*=0 \text{ on } \partial\Omega}{\mathrm{Min}} \frac{1}{2}\int_\Omega \mathbf{a}(\varepsilon(\mathbf{u}^*) - \underset{\sim}{\alpha})(\varepsilon(\mathbf{u}^*) - \underset{\sim}{\alpha})\,\mathrm{d}x - \int_\Omega \mathbf{f}\mathbf{u}^*\,\mathrm{d}x$$

The force $\underset{\sim}{\mathbf{A}}$ associated with $\underset{\sim}{\alpha}$ is easily identified as the field of the stress tensor. The class of materials obeying (57) reduces to elastic perfectly plastic bodies. The variational principle (70) is known as the Hodge–Prager variational principle [18], and since the operator

$$\frac{\partial^2 W^*}{\partial \underset{\sim}{\mathbf{A}}^2} = \mathbf{a}^{-1}$$

is positive definite, this principle ensures the existence and the uniqueness of the stress rate in a space of square integrable fields.

The variational principle (66) is known as Greenberg's principle [11]. However the involved operator

$$\frac{\partial^2 W}{\partial \underset{\sim}{\alpha}^2} \tag{71}$$

is not definite, and neither the existence nor the uniqueness of strain rate $\dot{\underset{\sim}{\alpha}} = (\dot{\varepsilon}^p)$ can be proved in a classical framework. Due to the lack of coercivity of the above mentioned operator (71) the functional (66) grows linearly with $\dot{\underset{\sim}{\alpha}}^*$ at infinity. Therefore its natural space of definition merely requires $\dot{\underset{\sim}{\alpha}}$ to be integrable and not square integrable, which would classically be the case in this kind of variational problem. In the case of perfect plasticity this remark has led a few authors [31, 32] to introduce in 1977 the space of vector fields with Bounded Deformation:

$$\mathrm{BD}(\Omega) = \{\mathbf{u} \mid \mathbf{u} = (u_i),\ u_i \in L^1(\Omega),\ \varepsilon_{ij}(\mathbf{u}) \in M^1(\Omega) 1 \leqslant i, j \leqslant 3\}$$

where $M^1(\Omega)$ is the space of bounded measures on Ω. BD(Ω) provides the good functional framework to prove the existence of a solution of the variational problem (66). One has to notice that elements of BD(Ω) in general, and solutions of (66) in particular, can be discontinuous fields even before the limit load of the structure is reached. This mathematical anomaly has been previously noticed through a completely different approach by Zyczkowski [38].

5. CONCLUSION

This paper intends to focus the attention on the progress achieved during the last decade in the field of mathematical plasticity. This progress results in a better understanding of the structure of the constitutive laws and of the mathematical properties of the related boundary value problems. Several open mathematical problems have been addressed which will probably be solved within the next ten years with the help of one of the tools presented here: homogenization, general standard materials, global variables.

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APPENDIX: HEGEMONY OF CONVEXITY?

This paper might appear to the reader as an anthem to convexity if we were not aware of the elementary and necessary restrictions to be brought into the theory.

(a) Convexity requires the space of states to be a *linear vector space*. Consequently it is inadequate for elasticity at finite strains: for instance, in an incompressible material the principal strains λ_i must satisfy

$$(1+\lambda_1)(1+\lambda_2)(1+\lambda_3)=1$$

which is a non-convex constraint, both on the λ_i and on the displacement field. Other incompatibilities (mainly with the frame indifference principle) have been pointed out by Hill [17]. However this argument fails in plasticity, which is concerned with strain rates, hence with linear functions of the velocity fields when computed in the correct configuration.

(b) Convexity is closely related to stability, each of these notions mainly containing the other one. On the one hand Drucker's work shows that material stability implies convexity of the yield surface and normality of the flow rule. On the other hand Hill [17] and Nguyen Quoc Son [24] established for a plastic material exhibiting a convex yield surface and obeying the normality rule, a stability criterion satisfied under reasonable assumptions for the hardening. Consequently micro and macro instabilities could need a non-convex investigation.

(c) Convexity is unable to account for the behavior of frictioning systems (Coulomb's law) and of soils. For materials obeying a non-associated flow rule Telega [36] used Sewell's account to extend the problem to a convex one and to establish variational principles. When the normality law fails to hold together with the convexity of the yield locus, other tools are to be developed: this was done by Salencon and Tristan Lopez [28] who extended the notion of limit analysis.

Finally we do not claim universal validity for convexity. But addressing the question 'convex or non-convex' defines the proper nature of convexity: it is a reference property.

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Inverse Problems in Structural Elastoplasticity: A Kalman Filter Approach

S. Bittanti[1], G. Maier[2] and A. Nappi[2]

[1] *Department of Electronics,* [2] *Department of Structural Engineering, Technical University (Politecnico), Milan, Italy*

ABSTRACT

Parameters governing local inelastic deformability in an elastic–plastic discrete structural model are determined, in the sense of Bayesian stochastic estimation, on the basis of information on meaningful displacements in the inelastic response of the structure to a given quasistatic loading history. This structural identification or 'inverse' problem is solved by an extended Kalman filter method. The formulation of the solution procedure consists of the following phases (corresponding to subsections of Section 3): state space representation of the model; linearization of the state equation; linearization of the output equation; weighted least square approach to the parameter estimation; recursive parameter estimation; extended Kalman filter equations; iterated extended Kalman filter. Numerical examples concerning frames illustrate and test the methodology adopted.

NOTATION

Bold-face symbols denote matrices (and column vectors). A tilde means transposed, $\mathbf{0}$ a matrix of all zero entries. Vector inequalities apply componentwise. The matrix of the derivatives of vector $\mathbf{y}$ with respect to $\mathbf{x}$ will be indicated by $\partial\mathbf{y}/\partial\tilde{\mathbf{x}}$, thus ordering columns according to the $\mathbf{x}$ components. The subscript t/τ means estimate of a given variable at time t based on data up to time τ.