Interface method and finite volumes: two-dimensional acoustic example

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ABSTRACT. *This paper is devoted to the numerical resolution of 2D acoustic wave propagation with fixed interfaces. Classical numerical schemes are used on a regular cartesian grid. However, properties of the schemes are damaged with interfaces: fall of the order, instabilities, numerical diffraction. To avoid these drawbacks, we propose to couple the schemes with an interface method, the Explicit Simplified Interface Method (ESIM), previously developed in 1D. This numerical method, based on the jump conditions at interfaces, can handle arbitrary-shaped interfaces immerged in the meshing. Numerical experiments are proposed, coupling a classical finite-volume scheme, the Wave Propagation Algorithm, with the ESIM.*

KEYWORDS: *Acoustic waves, discontinuous coefficients, interface methods, Wave Propagation Algorithm.*

1. Introduction

In this paper, we consider the numerical resolution of 2D linear hyperbolic problems (like acoustics) with fixed interfaces. To do so, we use finite-difference or finite-volume schemes, like the Wave-Propagation Algorithm developed by LeVeque [LEV 97], on a regular cartesian grid.This flux-limiter scheme owns useful properties for wave propagation simulations, like the truly multidimensional time-marching (reducing the numerical anisotropy induced by the grid), and it avoids spurious oscillations (induced by the numerical dispersion). However, qualities of this scheme are damaged in the presence of interfaces.

Indeed, interfaces introduce three kinds of difficulties for the numerical resolution. Firstly, the exact solution is not smooth across interfaces, which reduces the local truncation error and can develop instabilities for high contrasts of physical parameters. Secondly, arbitrary-shaped interfaces are poorly described by the regular cartesian grid: a “stair-step” description of interfaces introduces non-physical diffraction. Thirdly, physical jump conditions are ignored by the schemes.

The goal of our presentation is to propose a numerical method that eliminates these drawbacks. To do so, we develop an interface method, the Explicit Simplified Interface Method (ESIM), previously proposed in 1D [PIR01]. This method generalizes the classical interface methods like the Immersed Interface Method (IIM) [LI 94, ZHA97]

2. Presentation of the problem

Consider a perfect motionless fluid and a curve Γ $(x(\tau), y(\tau))$ dividing $\mathbb{R}^2$ in two subdomains Ω_0 and Ω_1. The density ρ and the sound speed c are discontinuous across Γ, and they are piecewise constant on both sides of Γ. Outside Γ, the acoustic equations are

$$\boxed{\begin{array}{l} \dfrac{\partial v_1}{\partial t} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0, \\[2ex] \dfrac{\partial v_2}{\partial t} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} = 0, \\[2ex] \dfrac{\partial p}{\partial t} + \rho\, c^2 \left(\dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y}\right) = 0, \end{array}} \tag{1}$$

where v_1, v_2 are the components of the acoustic velocity $\boldsymbol{v}$, and p is the acoustic pressure. Denoting the jump of a function f across Γ at P by

$$\begin{aligned} [f(P,t)] &= f(P^+,t) - f(P^-,t) \\ &= \lim_{M \to P, M \in \Omega_1} f(M,t) - \lim_{M \to P, M \in \Omega_0} f(M,t), \end{aligned} \tag{2}$$

the jump conditions at a point P of Γ are

$$\boxed{\begin{array}{l} [p(P,t)] = 0, \\[2ex] [v_N(P,t)] = 0, \end{array}} \tag{3}$$

whith $v_N = \boldsymbol{v}.\boldsymbol{n}$ and with $\boldsymbol{n}$ a unit normal vector at P on Γ.

3. Numerical schemes in homogeneous medium

Setting $\boldsymbol{U} = {}^T(v_1, v_2, p)$, we can write (1) as a first-order linear hyperbolic system outside Γ via some 3×3 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\frac{\partial}{\partial t}\boldsymbol{U} + \boldsymbol{A}\,\frac{\partial}{\partial x}\boldsymbol{U} + \boldsymbol{B}\,\frac{\partial}{\partial y}\boldsymbol{U} = \boldsymbol{0}. \tag{4}$$

To solve (4), we use two-step, explicit, and spatially-centered finite-volume schemes, like the Wave Propagation Algorithm developed by LeVeque [LEV 97]. Given a time step Δt and spatial mesh sizes $\Delta x = \Delta y$, we seek an average $\boldsymbol{U}_{i,j}^n$ of $\boldsymbol{U}(x, y, t_n)$ on the cell $[x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$. Time-marching at (i, j) is written symbolically

$$\begin{aligned}\boldsymbol{U}_{i,j}^{n+1} = \boldsymbol{U}_{i,j}^n \quad & - \quad \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}(\boldsymbol{U}^n; i, j) - \boldsymbol{F}(\boldsymbol{U}^n; i-1, j)\right) \\ & - \quad \frac{\Delta t}{\Delta y}\left(\boldsymbol{G}(\boldsymbol{U}^n; i, j) - \boldsymbol{G}(\boldsymbol{U}^n; i, j-1)\right),\end{aligned} \tag{5}$$

where $\boldsymbol{F}$ and $\boldsymbol{G}$ are numerical flux functions that depend on s numerical values surrounding (i, j), referred by their indices (i_l, j_l) $0 \leq l \leq s-1$. The scheme (5) is only applied at *regular points*, that is, at grid points where the stencil does not cross Γ. The time-marching at *irregular points* is described further.

4. The interface conditions

Let's introduce the notation

$$\boldsymbol{U_{ki}}(x, y, t) = {}^T\left(\boldsymbol{U}, \frac{\partial}{\partial x}\boldsymbol{U}, \ldots, \frac{\partial^\alpha}{\partial x^{\alpha-\beta}\,\partial y^j}\boldsymbol{U}, \ldots, \frac{\partial^k}{\partial y^k}\boldsymbol{U}\right) \tag{6}$$

($0 \leq \alpha \leq k$, $0 \leq \beta \leq \alpha$, $i = 0$ in Ω_0, $i = 1$ in Ω_1). Consider P on Γ: the goal of this section is to express $\boldsymbol{U_{k1}}(P^+, t)$ in terms of $\boldsymbol{U_{k0}}(P^-, t)$, and reciprocally, what is necessary for the interface method. The main difficulty is that there are less conditions at interfaces than components of $\boldsymbol{U_{ki}}$.

To get these conditions, we differentiate the jump conditions (3) k times in terms of t and τ. Time derivatives are replaced by spatial derivatives via the conservation laws (1). Since we have, for a sufficiently smooth function $f(x, y, t)$,

$$\forall t, \qquad \frac{\partial}{\partial \tau} f\left(P^\pm, t\right) = \frac{d\,x}{d\,\tau}\,\frac{\partial}{\partial x} f\left(P^\pm, t\right) + \frac{d\,y}{d\,\tau}\,\frac{\partial}{\partial y} f\left(P^\pm, t\right), \tag{7}$$

we can easily replace τ-derivatives of (3) by spatial derivatives. Note that the derivation of jump conditions introduces an insight on the geometry of Γ at P.

To complete the k-th order jump conditions, we use *compatibility conditions* [LOM 02]: since the fluid is irrotationnal, we get for all (x, y) and $1 \leq n \leq k$

$$\frac{\partial^n\, v_1}{\partial\, x^{n-i-1}\, \partial\, y^{i+1}} - \frac{\partial^n\, v_2}{\partial\, x^{n-i}\, \partial\, y^{i}} = 0, \qquad 0 \leq i \leq n-1. \tag{8}$$

The matrix relations deduced from the k-th order jump conditions and from the compatibility conditions are still underdetermined. To solve them, we perform a Singular Value Decomposition (SVD); then, the full span of solutions is summed up by

$$\boxed{\boldsymbol{U_{k1}}(P^+, t) = \boldsymbol{S} \begin{pmatrix} \boldsymbol{U_{k0}}(P^-, t) \\ \\ \boldsymbol{\Lambda} \end{pmatrix}}, \tag{9}$$

where $\boldsymbol{S}$ depends on the k-th order jump conditions and on the compatibility conditions, and $\boldsymbol{\Lambda}$ is a set of Lagrange multipliers.

5. The interface method

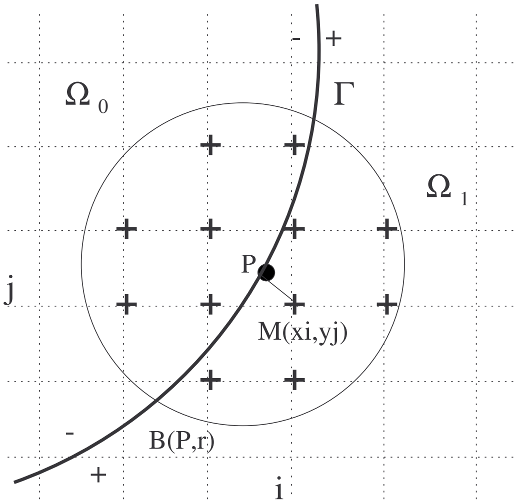

Figure 1. *2D interface* Γ*, irregular point* $M(x_i, y_j)$*, projection point* P*.*

Unlike classical interface methods, the ESIM uses the same numerical scheme everywhere. But the ESIM modifies the schemes implicitly at irregular points, by modifying the numerical values used for time-marching. Doing so, the modification of the scheme is transferred to the construction of *modified values*: as we show further, this is an easy and automatic task. The goal of this section is to describe how to compute and how to use these modified values.

At $t = t_n$, consider an irregular point $M(x_i, y_j)$ of Ω_1, and its orthogonal projection P on Γ (figure 1). Let's define the "Taylor matrix"

$$\boldsymbol{\Psi}_k(x,y) = \left(\boldsymbol{I}_3, (x-x_P)\,\boldsymbol{I}_3, (y-y_P)\,\boldsymbol{I}_3, \frac{1}{2}(x-x_P)^2\,\boldsymbol{I}_3, \ldots, (y-y_P)^k\,\boldsymbol{I}_3\right). \tag{10}$$

where $\boldsymbol{I}_3$ is the identity matrix 3×3. On Ω_1, we define a sufficiently smooth extension $\boldsymbol{U}^*(x, y, t_n)$ of $\boldsymbol{U}(x, y, t_n)$ on Ω_0, so that

$$\boldsymbol{U}^*(x_i, y_j, t_n) = \boldsymbol{\Psi}_k(x_i, y_j)\,\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n). \tag{11}$$

The numerical estimation of (11) at $M(x_i, y_j)$ is called a *modified value* and is denoted by $\boldsymbol{U}^*_{i,j}$. To estimate numerically $\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n)$ in (11), we use a set of q grid points surrounding P (denoted by '+' in figure 1) with coordinates $(x_{i_\theta}, y_{j_\theta})$. At $P^\pm$, we write the $(k+1)$-th order Taylor expansions of $\boldsymbol{U}(x_{i_\theta}, y_{j_\theta}, t_n)$. Hence, for $1 \leq \theta \leq q$,

$$\begin{aligned}
&(x_{i_\theta}, y_{j_\theta}) \in \Omega_0,\, \boldsymbol{U}(x_{i_\theta}, y_{j_\theta}, t_n) = \boldsymbol{\Psi}_k\,(x_{i_\theta}, y_{j_\theta})\;\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n) + \boldsymbol{O}(\Delta\,x^{k+1}),\\
&(x_{i_\theta}, y_{j_\theta}) \in \Omega_1,\, \boldsymbol{U}(x_{i_\theta}, y_{j_\theta}, t_n) = \boldsymbol{\Psi}_k\,(x_{i_\theta}, y_{j_\theta})\;\boldsymbol{U}_{\boldsymbol{k}\mathbf{1}}(P^+, t_n) + \boldsymbol{O}(\Delta\,x^{k+1}).
\end{aligned} \tag{12}$$

From conditions (9), we deduce

$$\begin{aligned}
&(x_{i_\theta}, y_{j_\theta}) \in \Omega_0,\, \boldsymbol{U}(x_{i_\theta}, y_{j_\theta}, t_n) = \boldsymbol{\Psi}_k\,(x_{i_\theta}, y_{j_\theta})\;\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n) + \boldsymbol{O}(\Delta\,x^{k+1}),\\
&(x_{i_\theta}, y_{j_\theta}) \in \Omega_1,\, \boldsymbol{U}(x_{i_\theta}, y_{j_\theta}, t_n) = \boldsymbol{\Psi}_k\,(x_{i_\theta}, y_{j_\theta})\;\boldsymbol{S}\begin{pmatrix}\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n)\\ \\ \boldsymbol{\Lambda}\end{pmatrix} + \boldsymbol{O}(\Delta\,x^{k+1}).
\end{aligned} \tag{13}$$

The compatibility conditions are used to decrease the number of unknowns $\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n)$ in the overdetermined system deduced from (13). Then, exact values $\boldsymbol{U}(x_{i_\theta}, y_{j_\theta}, t_n)$ are replaced by numerical values $\boldsymbol{U}^n_{i_\theta, j_\theta}$ $(1 \leq \theta \leq q)$, and truncation errors are eliminated, leading to least-squares estimations $\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}{}^-$ and $\tilde{\boldsymbol{\Lambda}}$ of $\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}(P^-, t_n)$ and $\boldsymbol{\Lambda}$. We can express $\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}{}^-$ via a matrix $\boldsymbol{M}^{-1}$

$$\boldsymbol{U}_{\boldsymbol{k}\mathbf{0}}{}^- = \boldsymbol{M}^{-1}\;{}^T\left(\boldsymbol{U}^n_{i_1,j_1}, \ldots \boldsymbol{U}^n_{i_q,j_q}\right). \tag{14}$$

From (11) and (14), we deduce the modified solution $\boldsymbol{U}^*_{i,j}$

$$\boxed{\boldsymbol{U}^*_{i,j} = \boldsymbol{\Psi}_k(x_i, y_j)\,\boldsymbol{M}^{-1}\begin{pmatrix}\boldsymbol{U}^n_{i_1,j_1}\\ \vdots \\ \boldsymbol{U}^n_{i_q,j_q}\end{pmatrix}.} \tag{15}$$

A similar procedure holds at each irregular points.

Modified values are used for time-marching at irregular points: at $M(x_i, y_j)$, the scheme uses numerical values at points on the same side of Γ than M (as usually), and modified values on the other side of Γ. To formalize this idea, we define the values $\tilde{\boldsymbol{U}}_{i_l,j_l}$ at points (x_{i_l}, y_{j_l}) $0 \leq l \leq s-1$

$$(x_{i_l}, y_{j_l}) \in \Omega_0 \Rightarrow \tilde{\boldsymbol{U}}_{i_l,j_l} = \boldsymbol{U}^*_{i_l,j_l}, \qquad (x_{i_l}, y_{j_l}) \in \Omega_1 \Rightarrow \tilde{\boldsymbol{U}}_{i_l,j_l} = \boldsymbol{U}^n_{i_l,j_l}. \quad (16)$$

Instead of (5), time-marching at the irregular point $M(x_i, y_j)$ is

$$\boxed{\begin{aligned} \boldsymbol{U}^{n+1}_{i,j} = \boldsymbol{U}^n_{i,j} \quad & - \quad \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}(\tilde{\boldsymbol{U}}^n; i, j) - \boldsymbol{F}(\tilde{\boldsymbol{U}}^n; i-1, j)\right) \\ & - \quad \frac{\Delta t}{\Delta y}\left(\boldsymbol{G}(\tilde{\boldsymbol{U}}^n; i, j) - \boldsymbol{G}(\tilde{\boldsymbol{U}}^n; i, j-1)\right). \end{aligned}} \quad (17)$$

Then, the scheme is said to be coupled with the ESIM. The value of k used in (12) depends on the order of the scheme. For a second-order scheme like the Wave Propagation Algorithm, we take $k = 2$. Since the interface does not move, the matrix $\boldsymbol{M}^{-1}$ need to be computed only once at each irregular point, during a preprocessing step. At each time step, only the matrix-vector multiplication (15) need to be done at each irregular point. Since the number of irregular points is much smaller than the number of grid points, the computational cost induced by the ESIM remains much smaller than that for time-marching. Note also that the ESIM does not depend on the expression of the scheme: so, the coupling with a high-order scheme is automatic.

6. Numerical experiments

The coupling of the Wave Propagation Algorithm and the ESIM is tested on an example. We consider a $L_x \times L_y = 0.2 \times 0.2$ m^2 fluid medium, with an inclined plane interface Γ (θ =70 degrees with the horizontal axis). The physical parameters are

$$(\rho, c) = \begin{cases} \rho_0 = 1000 \text{ kg/m}^3,\ c_0 = 1500 \text{ m/s if } M(x,y) \in \Omega_0, \\ \\ \rho_1 = 2000 \text{ kg/m}^3,\ c_1 = 3000 \text{ m/s if } M(x,y) \in \Omega_1. \end{cases} \quad (18)$$

The figure 2 shows the incident, reflected and transmitted plane waves at the initial time (a-b) and after 100 time steps: without the ESIM (c-d) and with the ESIM (e-f). In this last case, numerical diffractions are greatly reduced. We have also measured orders of convergence by refining the mesh. Thanks to the ESIM, the order is maintained, even in the presence of interfaces: 1.5 in norm L_∞, 2 in norm L_1. Lastly, we have considered cases of extrem contrasts in physical parameters (e.g. water-air). No instabilities were observed.

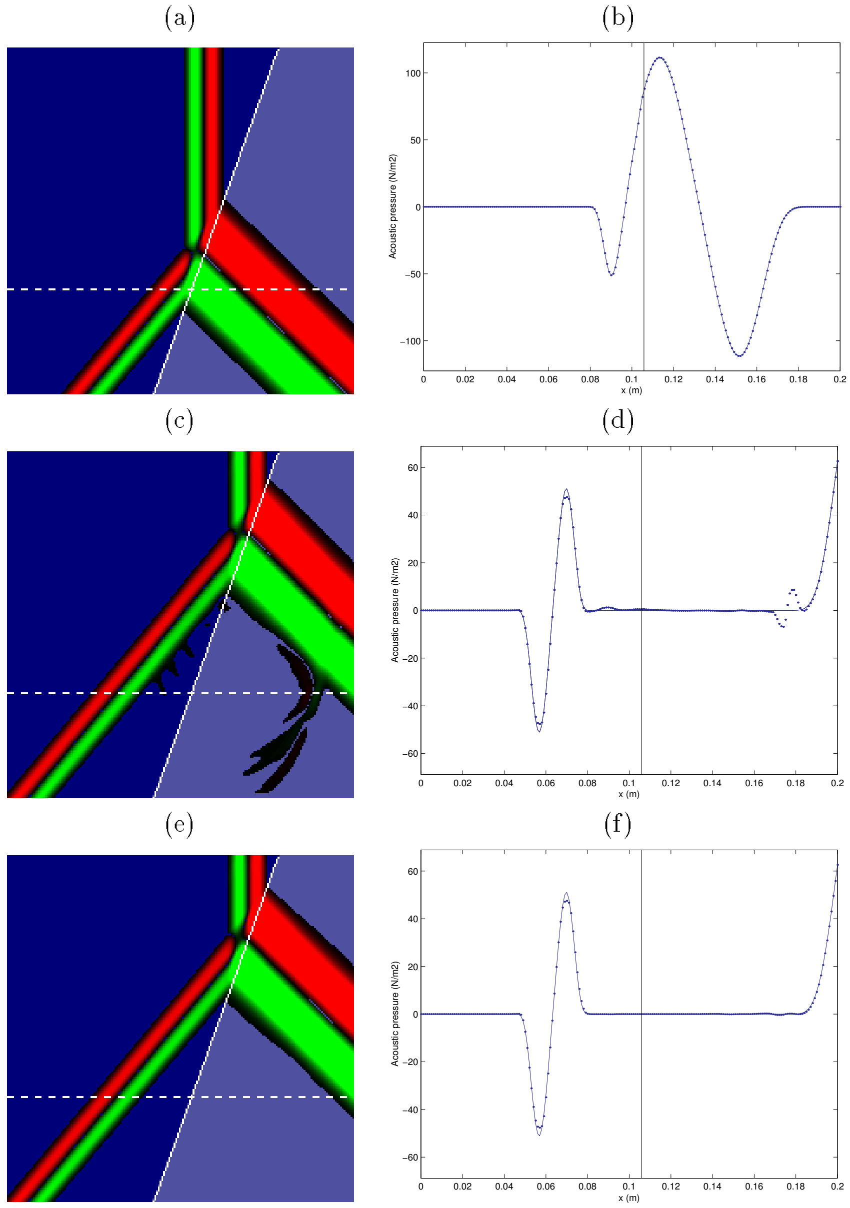

Figure 2. *Snapshots and slices of* p *along* y *=0.06 m (dotted horizontal line). Initial instant (a-b), after 100 time steps without (c-d) and with (e-f) the ESIM.*

7. Conclusion

We propose a numerical method that maintains, with interfaces, the properties of schemes in homogeneous medium. This interface method, the Explicit Simplified Interface Method (ESIM), incorporates the physical jump conditions in classical finite-difference or finite-volume schemes. It handles arbitrary-shaped interfaces on a regular grid. The ESIM has been coupled successfully with many schemes, like the Wave Propagation Algorithm.

Many points concerning the numerical analysis remain to be precised, and are currently under investigation. Firstly, we wish to demonstrate the stability of the ESIM (that has been verified numerically), at least when it is coupled with simple linear schemes. Secondly, we think that the method is conservative (at least up to a given order of accuracy): indeed, the ESIM enforces the numerical solution to satisfy the jump conditions, which are deduced from physical principles of conservation. This property should be important for further applications to non-linear cases, like Euler equations with obstacles.

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